

Mirror Symmetry for hyperkähler manifolds.

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We prove the Mirror Conjecture for Calabi-Yau manifolds equipped with a holomorphic symplectic form, also known as complex manifolds of hyperkähler type. We obtain that a complex manifold of hyperkähler type is mirror dual to itself. The Mirror Conjecture is stated (following Kontsevich, ICM talk) as the equivalence of certain algebraic structures related to variations of Hodge structures. We compute the canonical flat coordinates on the moduli space of Calabi-Yau manifolds of hyperkähler type, introduced to Mirror Symmetry by Bershadsky, Cecotti, Ooguri and Vafa.

1 Introduction.

By a holomorphically symplectic manifold we understand a complex manifold equipped with a closed holomorphic 2-form, which is non-degenerate. A hyperkähler manifold is a Riemannian manifold equipped with a quaternionic action which is parallel with respect to a Levi-Civita connection. Every hyperkähler manifold is complex (the complex structure is induced by any embedding $\mathbb{C} \hookrightarrow \mathbb{H}$) and holomorphically symplectic. The converse is also true in the compact case, as implied by the Calabi-Yau theorem. For details and basic results on hyperkähler manifolds, see [Bes], [V].

1.1 A summary for those who like physics.

Mirror Symmetry has a rich history, which this article mostly ignores. For up-to-date references to the physical literature, the reader is advised to look in [Mrr].

Let M be a compact holomorphically symplectic manifold. For a generic complex structure on M , M admits no holomorphic curves ([V-Sym]). This allows one to compute the (tautological, because no instanton corrections are required) correlation functions of the A-model, for M as a target space.

The moduli space $Comp$ of M is equipped with a locally biholomorphic map to a quadric hypersurface C in a complex projective space $\mathbb{P}H^2(M, \mathbb{C})$.

The space C is equipped with a transitive action of a group $G_0(M) \cong SO(3, n-3)$, $n = h^2(M)$. Consider the variation V of Hodge structures over $Comp$ associated with the cohomology of M . We prove that there exists a $G_0(M)$ -equivariant variation of Hodge structures \underline{V} on C such that V is a pullback of \underline{V} . This allows one to compute V explicitly. Using V , we compute the correlation functions for the B-model, in terms of the $G_0(M)$ -action.

Usually in Mirror Symmetry, the parameter space for the A-model is the complexified Kähler cone, but in the case of hyperkähler manifolds, the correlation functions can be analytically continued to the whole space $H^{1,1}(M)$, which contains the Kähler cone as an open subset. The parameter space for the B-model is the moduli space $Comp$, equipped with the canonical flat coordinates of [BCOV]. We identify these parameter spaces locally using the flat coordinates, and compare correlation functions, computed explicitly.

1.2 A summary for those who like mathematics.

Let M be a compact Calabi-Yau manifold.¹ Physicists associate with M two associative, graded commutative algebras with unit: the Yukawa algebra, determined by the complex structure, and the quantum cohomology algebra, which depends on the Kähler class. By definition, the Yukawa algebra of M is isomorphic to the ring $\oplus H^i(\Lambda^j TM)$, where $\Lambda^j TM$ is the j -th exterior power of the bundle of holomorphic vector fields, and multiplication is defined naturally by the Künneth formula (Section 4). Using the triviality of the canonical class of M , we obtain an isomorphism of holomorphic vector bundles

$$\eta : \Lambda^i TM \longrightarrow \Omega^{n-i} M,$$

where $n = \dim_{\mathbb{C}} M$. Thus, the Yukawa product can be considered as a multiplicative structure on the cohomology of M :²

$$\bullet_Y : H^{i,j}(M) \times H^{i',j'}(M) \longrightarrow H^{i+i'-n,j+j'}(M)$$

The quantum cohomology algebra (Definition 3.3) is a deformation of the usual cohomology algebra defined via counting the rational curves on M . As usual, an associative algebra is called **Frobenius** if it is equipped

¹In this article, by Calabi-Yau manifold we understand a Ricci-flat Kähler manifold.

²We denote by $\Omega^i(M)$ the sheaf of holomorphic i -forms on M . Thus, $H^i(\Omega^j(M)) = H^{j,i}(M)$.

with a non-degenerate invariant scalar product (Definition 2.1). The Yukawa algebra and the Quantum cohomology algebra are Frobenius, which follows from the definitions.

The main ingredient of the Mirror Conjecture is the existence of the so-called “Mirror dual” Calabi-Yau manifold W . Mirror Symmetry is often stated as an isomorphism between Frobenius algebras: the quantum cohomology Frobenius algebra associated with M is conjecturally isomorphic to the Yukawa cohomology Frobenius algebra of W , and vice versa. In this form, the Mirror Conjecture trivially holds for M and W being the same holomorphically symplectic manifold (Theorem 5.4).

The aim of this article is to prove the refined form of the Mirror Conjecture, which states an isomorphism between certain algebraic structures, which we call “variations of Frobenius algebras.”

A variation of Frobenius algebras (see Definition 2.5 for the *exact* definition) over a base X is a variation of Hodge structures B equipped with a structure of Frobenius algebra on its associated graded bundle B^{gr} , and a sheaf homomorphism $TX \xrightarrow{\tau} B^{gr}$, such that the Kodaira-Spencer map $B^{gr} \otimes TX \longrightarrow B^{gr}$ coincides with $x \otimes \vec{v} \longrightarrow x \cdot \tau(\vec{v})$. Every Calabi-Yau manifold M produces two variations of Frobenius algebras: the VFA of Yukawa, $Y(M)$ and the Quantum Cohomology³ VFA, $Q(M)$. In this setting, Calabi-Yau manifolds M and W are **Mirror dual** if the VFA $Q(M)$ is locally isomorphic to $Y(W)$ and $Y(M)$ is locally isomorphic to $Q(W)$, and these isomorphisms are compatible with flat coordinates on the moduli space of complex structures, introduced in [BCOV]. We give the precise statement of the Mirror Conjecture in Section 3. In a similar form, the Mirror Conjecture is stated by M. Kontsevich ([Ko]).

The Mirror Conjecture appears to be true for a wide variety of Calabi-Yau manifolds, but not for all Calabi-Yau. There is not a single case in which the Mirror Conjecture is proven for Calabi-Yau manifolds in the strict sense.⁴ There are several approaches to the Mirror Conjecture for K3 surfaces and compact tori, which are, by convention, Mirror self-dual: [Tod3], [BB], [AM] (for additional references see [AM]).

³Quantum VFA is defined only up to certain convergence assumptions, which are part of the Mirror Conjecture. These assumptions are satisfied for a compact holomorphically symplectic manifold which is generic in its deformation class.

⁴Calabi-Yau manifolds of dimension greater than 2 with holomorphic Euler characteristic $\chi(\mathcal{O}(M)) = 2$. In dimension 3, such manifolds were the original source and the testbed of the Mirror Conjecture.

1.3 On the equivariance of a variation of Hodge structures.

Let M be a holomorphic symplectic manifold of Kähler type, with

$$h^{2,0}(M) = 1, \quad h^1(M) = 0,$$

and $Comp$ be its (coarse, marked) moduli space. We have the **period map** $P_c : Comp \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ associating a line $H_I^{2,0}(M) \subset H^2(M, \mathbb{C})$ to a complex structure I . There is a canonical non-degenerate symmetric pairing

$$(\cdot, \cdot)_{\mathcal{H}} : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}$$

defined by Beauville (see [V], [V-a]). Complexifying $H^2(M, \mathbb{R})$, we can consider $(\cdot, \cdot)_{\mathcal{H}}$ as a complex-linear, complex-valued form on $H^2(M, \mathbb{C})$. For all $I \in Comp$, the point $P_c(I)$ belongs to the quadric cone $C \subset \mathbb{P}H^2(M, \mathbb{C})$,

$$C = \{l \mid (l, l)_{\mathcal{H}} = 0\}.$$

The Torelli principle (proved by Bogomolov, [B1]) implies that the map

$$P_c : Comp \rightarrow C$$

is étale.

Let $\underline{\mathcal{H}} = \oplus H^{p,q}(M)$ be the variation of Hodge structures (VHS) on $Comp$ associated with the total cohomology space of M . The Results of [V] imply that there exists a variation of Hodge structures \mathcal{H} on C , such that $\underline{\mathcal{H}}$ is the pullback of a variation of Hodge structures \mathcal{H} : $\underline{\mathcal{H}} = P_c^*(\mathcal{H})$ (also, this is an immediate implication of Proposition 10.3). The set C is equipped with a natural action of the group $G_0(M) = SO\left(H^2(M, \mathbb{R}), (\cdot, \cdot)_{\mathcal{H}}\right)$. The group $G_0(M)$ also acts in the total cohomology space $H^*(M)$ of M ([V]). The main idea used in the proof of Mirror Symmetry is the following theorem, implicit in [V]:

Theorem 1.1: The VHS \mathcal{H} is $G_0(M)$ -equivariant, under the natural action of $G_0(M)$ on C and \mathcal{H} .⁵

Proof: This is an immediate implication of Claim 10.5. ■

⁵The action of $G_0(M)$ on the trivial bundle $\mathcal{H} = H^*(M) \times C$ comes from a natural action of $G_0(M)$ on the space $H^*(M)$.

To make this statement more explicit, we recall that a variation of Hodge structures is a flat bundle over a complex manifold, equipped with a real structure and a holomorphic filtration (Hodge filtration), which is complementary to its complex conjugate filtration, and satisfies so-called **Griffiths transversality condition**. Then, Theorem 1.1 says that the action of $G_0(M)$ on \mathcal{H} maps flat sections to flat sections, and preserves the real structure and the Hodge filtration.

1.4 On the equivariance of the Yukawa multiplication.

Let \mathcal{H}_{gr} be associated graded bundle of the VHS \mathcal{H} ,

$$\mathcal{H}_{gr}\big|_{P_c(I)} = \oplus H_I^{p,q}(M).$$

Let K be the pullback of $\mathcal{O}(1)$ from $\mathbb{P}H^2(M, \mathbb{C})$ to $C \hookrightarrow \mathbb{P}H^2(M, \mathbb{C})$. Then Yukawa multiplication on \mathcal{H} is a map $\mathcal{H}_{gr} \times \mathcal{H}_{gr} \longrightarrow \mathcal{H}_{gr} \otimes K$, defined in the same way as the usual Yukawa product for the VHS associated with Calabi-Yau manifolds (see Section 2):

$$H_I^{p,q}(M) \times H_I^{p',q'}(M) \longrightarrow H_I^{p+p'-n, q+q'}(M) \otimes K\big|_I.$$

Since the VHS \mathcal{H} is equivariant, the bundle \mathcal{H}_{gr} is equipped with a natural $G_0(M)$ -equivariant structure. The bundle K is also naturally $G_0(M)$ -equivariant. The key theorem of this paper is the following.

Theorem 1.2: The Yukawa product $\mathcal{H}_{gr} \times \mathcal{H}_{gr} \longrightarrow \mathcal{H}_{gr} \otimes K$ is compatible with the $G_0(M)$ -equivariant structure in \mathcal{H}_{gr} , K .

Proof: This is Theorem 7.4 . ■

Theorem 1.2 explains how the Yukawa product varies with the variation of $x \in Comp$.

Theorem 1.2 is proved by the following argument. Let $n = \dim_{\mathbb{C}} M$. The holomorphic symplectic form Ω defines an identification

$$(\Omega^i(M))^* \cong \Omega^i(M). \quad (1.1)$$

The top exterior power of Ω is a non-degenerate section of the canonical class $\Omega^n(M)$, because Ω is symplectic. This section produces an isomorphism

$$(\Omega^i(M))^* \cong \Omega^{n-i}(M). \quad (1.2)$$

the composition of (1.1) and (1.2) gives an isomorphism

$$\eta : \Omega^i(M) \longrightarrow \Omega^{n-i}(M).$$

Now, η induces a natural isomorphism of linear spaces,

$$\eta : H^i(\Omega^j(M)) \longrightarrow H^i(\Omega^{n-j}(M)). \quad (1.3)$$

We call the map (1.3) **the Serre duality operator** (Section 9). By definition, the Yukawa product in cohomology of M coincides with the usual cup-product in cohomology twisted by η .

Let $\mathfrak{g}(M) \subset \text{End}(H^*(M))$ be the Lie algebra generated by the Hodge operators L_ω, Λ_ω , where ω runs through Kähler classes corresponding to all complex structures on M . Let $G(M) \subset \text{End}(H^*(M))$ be the Lie group associated with $\mathfrak{g}(M)$. We prove that η belongs to $G(M)$ (Theorem 9.1) and express η algebraically in terms of the holomorphic symplectic form (Lemma 10.4).

In [V], we construct $G_0(M) \subset \text{End}(H^*(M))$ as a subgroup of $G(M) \subset \text{End}(H^*(M))$. This gives a way to work with η in terms of the $G_0(M)$ -action. In particular, we obtain the following theorem, which easily implies Theorem 1.2.

Theorem 1.3: Let $(I, \Omega), (I', \Omega')$ be holomorphic symplectic structures on M , and $[\Omega], [\Omega'] \in H^2(M, \mathbb{C})$ be the corresponding cohomology classes. Let $g \in G_0(M)$ be a group element such that, under the natural action of $G_0(M)$ on $H^2(M)$, $g([\Omega]) = [\Omega']$. Let $\eta, \eta' \in \text{End}(H^*(M))$ be the Serre duality operators associated with $(I, \Omega), (I', \Omega')$. Then

$$g\eta g^{-1} = \eta'.$$

Proof: This is Claim 10.5. ■

1.5 On the Tian-Todorov coordinates.

Let $Comp$ be the (coarse, marked) moduli space of complex structures on a Calabi–Yau manifold M . The Bogomolov–Tian–Todorov theorem provides canonical flat coordinates on $Comp$, the so-called Tian-Todorov coordinates. We define these coordinates in Section 4. When M is a holomorphically symplectic manifold, it is possible to compute the Tian-Todorov coordinates explicitly.

The moduli space $Comp$ is equipped with a period map

$$Comp \longrightarrow \mathbb{P}H^2(M, \mathbb{C}),$$

which associates a line $l \in H^2(M, \mathbb{C})$, $l = H_I^{2,0}(M)$ to a complex structure $I \in Comp$. Let C be the set of all lines satisfying $(l, l)_{\mathcal{H}} = 0$. From the definition of the form $(\cdot, \cdot)_{\mathcal{H}}$ it follows that for all I , the periods of L lie in C . Let $P_c : Comp \longrightarrow C$ be period map. By Bogomolov and Beauville ([Beau]), the map P_c is étale. Thus, constructing local coordinates on $Comp$ is equivalent to constructing local coordinates on C .

In [V], we proved that the space $H^*(M)$ is equipped with a natural action of the Lie algebra $\mathfrak{g}_0(M) \cong \mathfrak{so}(H^2(M, \mathbb{R}), (\cdot, \cdot)_{\mathcal{H}})$. Let $G_0(M) \subset End(H^*(M))$ be the corresponding Lie group. We have shown ([V], Corollary 12.5) that $G_0(M)$ acts on the cohomology ring $H^*(M)$ by automorphisms. Clearly, $G_0(M) \otimes \mathbb{C}$ acts naturally on C . For $I \in Comp$, we denote by $ad I$ the endomorphism of $H^*(M)$ defined by $ad I(\omega^{p,q}) = \sqrt{-1}(p-q)\omega^{p,q}$, $\omega^{p,q} \in H^{p,q}(M)$. In [V], we show that $ad I$ belongs to $\mathfrak{g}_0(M) \subset End(H^*(M))$ ([V], Theorem 12.2).

Every complex structure $I \in Comp$ defines a decomposition of the Lie algebra

$$\mathfrak{g}_0(M) = \mathfrak{g}_0^{I,-2}(M) \oplus \mathfrak{g}_0^{I,0}(M) \oplus \mathfrak{g}_0^{I,2}(M)$$

with

$$\mathfrak{g}_0^{I,i}(M) = \{x \in \mathfrak{g}_0(M) \mid [ad I, x] = i\sqrt{-1}x\}.$$

Let $G_0^{I,i}(M) \subset End(H^*(M))$ be the Lie group associated with $\mathfrak{g}_0^{I,i}(M)$. Then $G_0^{I,0}(M)$ and $G_0^{I,2}(M)$ stabilize $P_c(I) \in C$. Let $\varphi : G_0^{I,-2}(M) \longrightarrow C$ map $\alpha \in G_0^{I,-2}(M)$ to $\alpha(P_c(I)) \in C$. The group $G_0(M) \otimes \mathbb{C}$ transitively acts on C . Comparing dimensions of C and $G_0^{I,-2}(M)$, we find that φ is an isomorphism locally in a neighbourhood of the identity. Since the group $G_0^{I,-2}(M)$ is abelian, φ defines local flat coordinates in C .⁶ Since P_c is étale, φ also gives local coordinates on $Comp$. In Section 11, we prove that these coordinates coincide with the Tian-Todorov coordinates.

⁶Throughout this paper, the flat coordinates in a neighbourhood U of $x \in X$ are understood as a set S of pairwise commuting linearly independent holomorphic vector fields, $|S| = \dim_{\mathbb{C}} X$, defined in U . We say that X is **equipped with flat coordinates** if each point x of X has a neighbourhood equipped with such set S_x of holomorphic vector fields. No conditions of compatibility between S_x , for different x , is required.

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12 Proof of Mirror Conjecture.

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- The Introduction (Section 1) is included to give an outline of our reasoning. Section 1 refers to the body of the article for proofs, but otherwise, Section 1 and the rest of the paper are completely independent.
 - Section 2 gives a number of definitions culminating in the definition of variation of Frobenius algebras (VFA). We give an example of VFA (so-called **Yukawa VFA**), which is associated with every Calabi-Yau manifold.
 - Section 3 provides another example of variation of Frobenius algebras (so-called **Quantum Cohomology VFA**), conjecturally associated with every Calabi-Yau manifold. The existence of Quantum Cohomology VFA is a prerequisite for existence of the Mirror Dual manifold. The existence of Quantum Cohomology VFA is proven for holomorphically symplectic manifolds of generic type. In Section 3 we follow [KM], [Ko].
 - Section 4 gives proofs of a number of basic results on deformations of Calabi-Yau manifolds, mostly due to Bogomolov, Tian and Todorov ([B1] [Ti], [Tod1]). Section 4 is independent of Sections 2–3, and uses the basics of algebraic geometry and deformation theory ([GH], [KS]). As a final result, we obtain a definition of canonical flat coordinates on the moduli space of Calabi-Yau manifolds. These coordinates were introduced to Mirror Symmetry by [BCOV]. The exposition follows [Tod1].
 - Section 5 gives a statement of the Mirror Conjecture following [Ko], [BCOV]. There are no original results in Sections 2–5. Also, nothing in Sections 2–5 is in any way specific to hyperkähler or holomorphically symplectic geometry. We are dealing with manifolds of hyperkähler type only starting from Section 6.
 - Section 6 gives an explicit description of Quantum VFA for a holomorphically symplectic manifold which is generic in its deformation class. In [V-Sym], we proved that such manifolds have no rational curves. Since the “instanton corrections” (non-trivial terms of the Quantum product on cohomology) are expressed by counting rational curves,

it is easy to give an explicit description of Quantum VFA when all rational curves are trivial.

- Section 7 gives an outline of our description of the Yukawa VFA $\underline{\mathcal{A}}$ for a holomorphically symplectic manifold, with most proofs postponed till Section 10. Let C be the period space of M , and $Comp$ be the (coarse, marked) moduli space of M . By Bogomolov, the period map $Comp \rightarrow C$ is étale. In Section 7 we explain how $\underline{\mathcal{A}}$ is related to the period space C . We identify C with a certain orbit in the adjoint representation of the group $SO(m-3, 3)$, where $m = h^2(M)$. We then construct an $SO(m-3, 3)$ -equivariant VFA \mathcal{A} on C (existing modulo Theorem 7.3, which is proven in Section 10). Finally Theorem 7.4, proven in Section 10, states that $\underline{\mathcal{A}}$ is a pullback of \mathcal{A} under the period map.
- Sections 8 – 10 are dedicated to the proof of results outlined in Section 7. These sections are independent of the first part of this paper, but rely heavily on [V] .
- Section 8 deals with a linear-algebraic structure of the exterior algebra $\Lambda^*(T)$ of a quaternionic-Hermitian space T . We explain how results about $\Lambda^*(T)$ imply statements about the cohomology of a hyperkähler manifold. We construct an action of the group $Spin(4, 1)$ on the cohomology of a hyperkähler manifold and explicitly describe the action of its center.
- Section 9 uses results of Section 8 to describe the operator of Serre duality η . We prove that η belongs to the group $Spin(4, 1)$ acting on the cohomology of M . The operator η is expressed explicitly via the natural realization of $Spin(4, 1) = Sp(1, 1)$ in $End_{\mathbb{H}}(\mathbb{H}^2)$
- In Section 10 we apply the results of Section 9 to prove Theorem 7.3, Theorem 7.4. We prove the key results about the equivariant structure on the variations of Hodge structures corresponding to the cohomology of a holomorphically symplectic manifold.
- Section 11 computes the Tian-Todorov coordinates on $Comp$ in terms of the period map, for a holomorphically symplectic manifold. We use the results of [V], and the exact form of Tian-Todorov coordinates as given in Section 4.

- Section 12 gives a proof of the Mirror Conjecture for holomorphically symplectic manifolds. We rely heavily on results obtained in Sections 6 – 11.

2 Variations of Frobenius algebras.

Definition 2.1: (Frobenius algebras) A Frobenius algebra over a base field k is a k -linear space equipped with a structure of associative algebra and a linear map $\varepsilon : A \rightarrow k$, such that the bilinear form $(\cdot, \cdot)_A : a, b \rightarrow \varepsilon(ab)$ is non-degenerate. Let $(\cdot, \cdot, \cdot)_A : a, b, c \rightarrow \varepsilon(abc)$ be the trilinear form associated with the multiplicative structure in A . The pair $((\cdot, \cdot)_A, (\cdot, \cdot, \cdot)_A)$ is called a Frobenius structure on A . Clearly, the knowledge of $((\cdot, \cdot)_A, (\cdot, \cdot, \cdot)_A)$ suffices to recover the product in A . The linear map $\varepsilon : A \rightarrow k$ is often called **the trace form** of A . The Frobenius algebra A is called **graded** if A is equipped with a grading, which is respected by the multiplication, and there is a number n such that $A_n \cong k$ and the map ε factors through the natural projection $A \rightarrow A_n$.

Definition 2.2: Let X be a complex variety and $B = B_0 \oplus B_1 \oplus \dots \oplus B_n$ be a graded holomorphic vector bundle over X , equipped with a non-degenerate holomorphic pairing $(\cdot, \cdot) : B \times B \rightarrow \mathcal{O}_X$ and a holomorphic 3-form $(\cdot, \cdot, \cdot) : B \times B \times B \rightarrow \mathcal{O}_X$. Assume that $(\cdot, \cdot), (\cdot, \cdot, \cdot)$ define a structure of graded Frobenius algebra on the fibers of B in every point of X . Let $\varphi : TX \hookrightarrow B$ be a morphism of holomorphic vector bundles. Then $(B, \varphi, (\cdot, \cdot), (\cdot, \cdot, \cdot))$ is called **a weak variation of Frobenius algebras over X** , or simply **weak VFA**.

Definition 2.3: A weak complex variation of Hodge structures (or, weak \mathbb{C} -VHS) is the following collection of data.

1. A vector bundle B over a complex manifold X .
2. A flat connection $\nabla : B \rightarrow B \otimes \Lambda^1(X)$ on B . As usual, the flat connection induces a holomorphic structure on B .
3. A system of holomorphic subbundles $B^0 \subset B^1 \subset \dots \subset B_n = B$, which satisfies $\nabla B_i \subset B_{i+1} \otimes \Lambda^1 X$.

The filtration $B^0 \subset B^1 \subset \dots \subset B_n = B$ is called **the Hodge filtration** of the weak \mathbb{C} -VHS B .

Definition 2.4: Let $(B, \nabla, B^0 \subset B^1 \subset \dots \subset B_n = B)$ be a weak \mathbb{C} -VHS over a complex manifold X . On the associated graded factors, connection ∇ induces a holomorphic map

$$KS : B_i/B_{i-1} \longrightarrow (B_{i+1}/B_i) \otimes_{\mathcal{O}_X} \Omega^1 X. \quad (2.1)$$

This map is called **the Kodaira-Spencer map associated with the weak \mathbb{C} -VHS**. The map (2.1) induces a holomorphic map

$$B_{gr} \longrightarrow B_{gr} \otimes_{\mathcal{O}_X} \Omega^1 X, \quad (2.2)$$

where $B_{gr} = \oplus B_i/B_{i-1}$ is the associated graded quotient of B . The map (2.2) is called **the Higgs field**, or **the Kodaira-Spencer map** associated with the weak \mathbb{C} -VHS.

Definition 2.5: Let X be a complex manifold. A variation of Frobenius algebras (VFA) on X is a weak C -VHS $(B, \nabla, B^0 \subset B^1 \subset \dots \subset B^n = B)$, equipped with the following data.

- 0. A flat decomposition $B = B^{\text{odd}} \oplus B^{\text{even}}$, which is compatible with the weak \mathbb{C} -VHS structure. Let

$$(B^0)^{\text{even}} \subset (B^1)^{\text{even}} \subset \dots \subset (B^n)^{\text{even}} = B^{\text{even}},$$

$$(B^0)^{\text{odd}} \subset (B^1)^{\text{odd}} \subset \dots \subset (B^n)^{\text{odd}} = B^{\text{odd}},$$

be the Hodge filtration on $B^{\text{even}}, B^{\text{odd}}$. We denote $(B^i)^{\text{even}}/(B^{i-1})^{\text{even}}$ by A_{2i} , $(B^i)^{\text{odd}}/(B^{i-1})^{\text{odd}}$ by A_{2i+1} . The bundle $A = \oplus A_i$ is naturally isomorphic to B_{gr} .

- 1. An isomorphism $\mathcal{O}_X \cong (B^0)^{\text{even}}$.
- 2. A structure of weak VFA on $A = \oplus A_i$, which satisfies the following condition **•**. Let $\tau : TX \longrightarrow A$ be the map associated with the weak VFA structure, $\underline{t} : A \otimes TX \longrightarrow A$ be the homomorphism mapping $a \otimes \overline{v}$ to $a \cdot \tau(\overline{v})$ and $t : A \longrightarrow A \otimes \Omega^1 X$ the map obtained by the duality $(TX)^* \cong \Omega^1 X$.
- Under the natural identification $A \cong B_{gr}$, the map t corresponds to the Higgs field (a. k. a. Kodaira-Spencer map).

Example 2.6: (Yukawa VFA) Let M be a Calabi-Yau manifold, X its (coarse, marked) moduli space, and K be the standard line bundle on X with $K|_I = H_I^{n,0}(M)$ (sometimes called **the determinant bundle**). Let $\theta : \mathcal{O}_U \xrightarrow{\sim} K$ be a holomorphic trivialization of K over an open set $U \hookrightarrow X$. Let B be a VHS over X corresponding to the total cohomology space of M , and B^{gr} be the associated graded vector bundle. To introduce a VFA on B , we need to define an algebraic structure on B^{gr} . Let $n = \dim_{\mathbb{C}} M$. The trivialization $\theta : \mathcal{O}_U \xrightarrow{\sim} K$ identifies the sheaf of holomorphic i -forms $\Omega^i(M)$ with $\Lambda^{n-i}TM$, where TM is the holomorphic tangent bundle (see Section 4.1 for details of this identification). Since $H^{p,q}(M) = H^q(\Omega^p(M))$, the isomorphism $\Omega^i(M) \cong \Lambda^{n-i}TM$ results in a natural isomorphism $H^i(\Omega^{n-i}(M)) \cong H^i(\Lambda^i TM)$. The direct sum $\oplus_i H^i(\Lambda^i TM)$ is equipped with a natural multiplicative structure. This gives a multiplication on $B = \oplus H^{n-i,i}(M) = \oplus H^i(\Omega^{n-i}(M))$. The 2-form (\cdot, \cdot) on B comes from the Poincaré pairing. Finally, by Tian-Todorov, $T_I X$ is canonically isomorphic to $H^1(TM_I) \cong H_I^{n-1,1}(M)$. This gives an embedding $TM \hookrightarrow B$. Compatibility of these data and VHS (condition \bullet of Definition 2.5) is a standard result which follows from Kodaira-Spencer theory. The obtained VFA is called **the Yukawa VFA associated with the trivialization $\theta : \mathcal{O}_U \xrightarrow{\sim} K$** .

3 Dubrovin algebras and quantum VFA.

In this section, we construct a Mirror counterpart to the Yukawa variation of Frobenius algebras, which is called **quantum VFA**. For a graded linear space A , let $\text{Aff}(A)$ be an affine supermanifold which corresponds to A .

Definition 3.1: [KM] Let A be a graded linear space,¹ and $S = \text{Aff}(A)$. Let g_{ef} be a standard pairing on A considered as a constant bilinear form on S and v_0 be an even vector field on S (called **the unit field**) in A . Let Φ be a function on S (defined on all of S or, as in many geometrical cases, on its open subset) which satisfies the following assumptions.

(i) $v_0(\Phi) = \Phi$

$$(ii) \quad \sum_{e,f} \frac{\partial^3 \Phi}{\partial_a \partial_b \partial_e} g^{ef} \frac{\partial^3 \Phi}{\partial_f \partial_c \partial_d} = (-1)^{\deg a (\deg b + \deg c)} \sum_{e,f} \frac{\partial^3 \Phi}{\partial_b \partial_c \partial_e} g^{ef} \frac{\partial^3 \Phi}{\partial_f \partial_d \partial_a}.$$

¹In applications, A is a space underlying the cohomology algebra

- (iii) For all $s \in S$, the vector $v_0|_s \in T_s S$ is a unit in the algebra $T_s S$ defined by (ii).

The pair (A, Φ) is called a **Dubrovin algebra**, and a function Φ is called **Dubrovin potential**.

Dubrovin algebras have the following properties.

Proposition 3.2: [KM] Let ∇_0 be a trivial flat connection on the tangent bundle to S . Let $C_{b,c}^a \in S^2 \Lambda^1 S \otimes TS$ be the tensor obtained from $\frac{\partial^3 \Phi}{\partial_c \partial_b \partial_e}$ by pairing with g :

$$C_{b,c}^a = \sum_{d,e} \frac{\partial^3 \Phi}{\partial_a \partial_b \partial_d} g_{d,e}$$

We consider $C_{b,c}^a$ as 1-form with coefficients in $\text{End}(TS)$. Then

- (i) For all t , the operator $\nabla_t = \nabla_0 + C_{b,c}^a$ is a flat connection on the tangent bundle to S .
- (ii) For all $s \in S$, the tensor $C_{b,c}^a$ and the form $g_{e,f}$ define a structure of Frobenius algebra on $T_s S$.

■

Let V be a compact algebraic (or symplectic, or Kähler) manifold. In [KM],² Kontsevich and Manin define, axiomatically, systems of classes

$$I_{g,n;\beta} \in H_*(\bar{M}_{g,n}) \otimes (H_*(V)^{\otimes n}),$$

where $\bar{M}_{g,n}$ is the Deligne-Mumford compactification of the space of curves of genus g with n marked points. These homology classes are called **Gromov-Witten classes**. For a system of Gromov-Witten classes, Kontsevich-Manin write down a power series

$$\Phi_\omega(\gamma) := \sum_{\beta \in H_2(V, \mathbb{Z})} e^{-\int_\beta \omega} \sum_{n \geq 3} \frac{1}{n!} \int_{I_{0,n;\beta}} 1_{\mathcal{M}_{0,n}} \otimes \gamma \otimes \cdots \otimes \gamma. \quad (3.1)$$

depending on a parameter $\omega \in H^{1,1}(V)$, with an argument $\gamma \in H^*(M)$ (see [KM] for details of this definition). If this series converges, [KM] prove that it converges to a Dubrovin potential for a Frobenius algebra $A = \oplus H^{p,p}(V)$.

²In this account, we follow [Ko], which differs from [KM] in details.

The Gromov-Witten classes are proven to exist when V is a compact algebraic manifold ([Ko]). This statement automatically carries over to Kähler manifolds due to results about compactness of Chow schemes for Kähler manifolds (see, e. g., [Li]).³ The resulting system of Gromov-Witten classes coincides with the algebraic one when V is algebraic.

The power series (3.1) is known to converge when V is rational, and conjecturally also converges (at least in some subset of S) for V Calabi-Yau. This convergence is a part of the Mirror Conjecture. The Dubrovin potential $\Phi_\omega(\gamma)$ obtained this way is called **the quantum cohomology potential**. Let $U \subset H^{1,1}(V)$ be the domain of convergence for $\Phi_\omega|_{\mathfrak{B} \times H^{1,1}(V)}$ considered as a function of ω , with γ running through an infinitesimal ball \mathfrak{B} in $H^*(V)$ with center in 0.

Definition 3.3: Consider the Frobenius algebra structure defined on the space $H^*(V)$ by the 3-form $\frac{\partial^3 \Phi_\omega}{\partial_a \partial_b \partial_e} \Big|_0$, and the 2-form which is Poincaré pairing. Thus obtained Frobenius algebra is called **the quantum cohomology ring of V , associated with $\omega \in H^{1,1}(V)$** .

Let \mathcal{A} be a trivial bundle over U , with a fiber $H^*(V)$. Dubrovin potential (3.1) defines a structure of Frobenius algebra on \mathcal{A} , with multiplication in $\mathcal{A}|_\omega$, $\omega \in U$ defined by the tensor

$$C_{a,b}^c(\omega) = \frac{\partial^3 \Phi_\omega}{\partial_a \partial_b \partial_e} \Big|_0 g^{ec}, \quad (3.2)$$

where $g^{ef} \in H^*(V) \otimes H^*(V)$ is the 2-vector defined by the Poincaré pairing. This is just another version of Definition 3.3.

Let ∇_0 be a trivial connection in \mathcal{A} , and $\nabla_m := \nabla_0 + C$ be the connection defined by the map $C|_\omega : T_\omega U \longrightarrow \text{End}(H^*(V))$, with $C(t)(\alpha) = C_{a,b}^c(\omega)(t, \alpha)$ for all $t \in T_\omega U = H^{1,1}(V)$, $\alpha \in H^*(V) = \mathcal{A}|_\omega$.⁴ Let $\mathcal{A}_i \subset \mathcal{A}$ be the constant sub-bundle with a fiber $H^i(V) \subset H^*(V) = \mathcal{A}|_\omega$. Applying the renumbering procedure of Definition 2.5 **0.** backwards, we obtain a decomposition $\mathcal{A} = \mathcal{A}^{\text{even}} \oplus \mathcal{A}^{\text{odd}}$ and a filtration $\mathcal{A}_0^{\text{even}} \subset \mathcal{A}_1^{\text{even}} \subset \dots$,

³However, in the case we are considering throughout this paper, V is a holomorphically symplectic manifold without holomorphic curves, so Gromov-Witten classes are tautological, and (3.1) converges automatically for all γ .

⁴In a more general setting, connections ∇_0, ∇_m were considered in Proposition 3.2; in particular, Proposition 3.2 immediately implies that ∇_m is flat.

$\mathcal{A}_0^{\text{odd}} \subset \mathcal{A}_1^{\text{odd}} \subset \dots$, on each of these bundles. This constructed filtration is obviously constant with respect to ∇_0 . Therefore, \mathcal{A} is naturally isomorphic to its associated graded quotient \mathcal{A}^{gr} . We obtain that the multiplication (3.2) is naturally defined on \mathcal{A}^{gr} . We denote it by $\bullet : \mathcal{A}^{gr} \times \mathcal{A}^{gr} \longrightarrow \mathcal{A}^{gr}$.

Proposition 3.4: (Kontsevich, [Ko]) Let V be a manifold equipped with a system of Gromov–Witten classes $I_{g,n;\beta}$. Assume that $c_1(V) = 0$ and for all n, β , the dimension of $I_{0,n;\beta}$ is the minimum value predicted by the Atiyah–Singer theorem:

$$\dim_{\mathbb{C}} I_{0,n;\beta} = n + \dim_{\mathbb{C}} V - 3.$$

Let $U \subset H^{1,1}(V)$ be a domain of convergence for (3.2), and \mathcal{A} be a trivial bundle over U with a fiber $H^*(V)$, equipped with a decomposition

$$\mathcal{A} = \mathcal{A}^{\text{even}} \oplus \mathcal{A}^{\text{odd}},$$

a filtration

$$\mathcal{A}_0^{\text{even}} \subset \mathcal{A}_1^{\text{even}} \subset \dots, \quad \mathcal{A}_0^{\text{odd}} \subset \mathcal{A}_1^{\text{odd}} \subset \dots, \quad (3.3)$$

$\mathcal{A}_k^{\text{even}}|_{\omega} = \bigoplus_{i=0}^k H^{2i}(V)$, $\mathcal{A}_k^{\text{odd}}|_{\omega} = \bigoplus_{i=0}^k H^{2i+1}(V)$, and connections ∇_0, ∇_m as above. Then

$$\begin{aligned} \nabla_m(\mathcal{A}_k^{\text{even}}) &\subset \mathcal{A}_{k+1}^{\text{even}} \otimes \Omega^1 U \\ \nabla_m(\mathcal{A}_k^{\text{odd}}) &\subset \mathcal{A}_{k+1}^{\text{odd}} \otimes \Omega^1 U \end{aligned} \quad (3.4)$$

Remark 3.5: From (3.4) it follows immediately that the filtration (3.3) and the connection ∇_m define on the bundles $\mathcal{A}^{\text{even}}, \mathcal{A}^{\text{odd}}$ a complex variation of Hodge structures.

Proof: Clearly, $\nabla_0(\mathcal{A}_k^{\varepsilon}) \subset \mathcal{A}_k^{\varepsilon} \otimes \Omega^1 U$, for $\varepsilon = \text{even, odd}$. By definition of ∇_m , to prove (3.4), we need only to show that

$$C(\mathcal{A}_k^{\varepsilon}) \subset \mathcal{A}_{k+1}^{\varepsilon} \otimes \Omega^1 U, \quad \varepsilon = \text{even, odd} \quad (3.5)$$

Unraveling the definition of C , we obtain that (3.5) is implied by the following lemma.

Lemma 3.6: Let $c \in H^{1,1}(V)$, $\omega \in U$, and

$$\bullet_\omega : H^*(V) \times H^*(V) \longrightarrow H^*(V)$$

be the quantum multiplication on $H^*(M)$ associated with ω (see Definition 3.3). Then $c \bullet_\omega \alpha \in H^{i+2}(V)$, for all $\alpha \in H^i(V)$.

Proof: Let

$$(\cdot, \cdot) : H^*(V) \times H^*(V) \longrightarrow \mathbb{C}$$

be the Poincaré form, and

$$(x, y, z)_c : H^*(V) \times H^*(V) \times H^*(V) \longrightarrow \mathbb{C}$$

be the form defined by $(x, y, z)_c = (x \bullet_\omega y, z)$. Let

$$J_{g,n;\beta} \in (H^*(V, \mathbb{Q}))^{\otimes n} \longrightarrow H^*(\bar{M}_{g,n}, \mathbb{Q})$$

be the map defined by the Poincaré duality from a cycle

$$I_{g,n;\beta} \in (H_*(V, \mathbb{Q}))^{\otimes n} \otimes H_*(\bar{M}_{g,n}, \mathbb{Q}).$$

Since $\bar{M}_{0,3}$ is a point, we have that $H_*(\bar{M}_{0,3}, \mathbb{Q}) = \mathbb{Q}$. Therefore, we may consider $J_{0,3,\beta}$ as a map from $(H^*(V, \mathbb{Q}))^{\otimes 3}$ to \mathbb{Q} . From (3.1), we have

$$(c \bullet_\omega \alpha, \alpha') = \sum_{\beta \in H^2(V, \mathbb{Z})} e^{-\int_V \beta} \frac{1}{3!} J_{0,3,\beta}(c, \alpha, \alpha'). \quad (3.6)$$

By the assumptions of Proposition 3.4, $\dim I_{0,3,\beta} = \dim V$. Therefore,

$$J_{0,3,\beta}(c, \alpha, \alpha') = 0$$

unless

$$\dim c + \dim \alpha + \dim \alpha' = 2 \dim_{\mathbb{R}} V.$$

On the other hand, the Poincaré form is non-zero only on cocycles of complementary dimension. Therefore, $\dim(c \bullet_x \alpha) = \dim \alpha + 2$. This proves Proposition 3.4. ■

Definition 3.7: Under the assumptions of Proposition 3.4, let

1. ∇_m ,
2. $\mathcal{A} = \mathcal{A}^{\text{even}} \oplus \mathcal{A}^{\text{odd}}$,

$$3. \mathcal{A}_0^{\text{even}} \subset \mathcal{A}_1^{\text{even}} \subset \dots, \mathcal{A}_0^{\text{odd}} \subset \mathcal{A}_1^{\text{odd}} \subset \dots,$$

$$4. \bullet : \mathcal{A}^{gr} \times \mathcal{A}^{gr} \longrightarrow \mathcal{A}^{gr}$$

be the structures we defined above. Clearly, **1 - 4** define a VFA on \mathcal{A} . The so-defined VFA is called **the Quantum VFA (or Quantum Deformation VFA)** associated to the system of Gromov–Witten classes on V .

Remark 3.8: For V a complex manifold with no holomorphic curves, the assumptions of Proposition 3.4 are tautologically satisfied. Therefore, for such V the Quantum Deformation VFA is defined correctly.

4 Tian–Todorov coordinates.

In this section, we define the canonical coordinates on the moduli space of Calabi-Yau manifolds ([BCOV]), to use further in the definition of the Mirror Symmetry.

4.1 Yukawa map.

Let M be an n -dimensional Calabi-Yau manifold equipped with a Ricci-flat metric and a non-degenerate section of the canonical class. Let $\Lambda^i TM$ be the i -th exterior power of the holomorphic tangent bundle to M . Clearly, $\Lambda^i TM$ is dual to the bundle $\Omega^i M$ of holomorphic differential i -forms. On the other hand, a non-degenerate section Ω of the canonical class defines a duality between $\Omega^i M$ and $\Omega^{n-i} M$. Thus, we obtain a natural isomorphism $\iota : \Lambda^i TM \longrightarrow \Omega^{n-i} M$. Since M is Ricci-flat, Ω is parallel by Bochner-Lichnerowicz. Therefore, the map ι is compatible (up to a coefficient) with the natural Hermitian structures on the bundles $\Lambda^i TM, \Omega^{n-i} M$. The isomorphism ι induces an isomorphism of cohomology

$$\eta : H^*(\Lambda^i TM) \xrightarrow{\sim} H^*(\Omega^{n-i} M)$$

which is called **the Yukawa map**. Consider the map of differential forms with values in vector bundles,

$$\tilde{\eta} : \Lambda^*(\Lambda^i TM) \longrightarrow \Lambda^*(\Omega^{n-i} M) \quad (4.1.1)$$

Since η is compatible with the metric, η maps harmonic forms to harmonic forms. Hence, the Yukawa map is induced by $\tilde{\eta}$. Further on, we don't discriminate between η and $\tilde{\eta}$.

4.2 Kodaira–Spencer theory.

Let $[\cdot, \cdot] : TM \times TM \longrightarrow TM$ be the commutator bracket. Consider the corresponding bracket on the cohomology

$$[\cdot, \cdot] : H^1(\Lambda^1 TM) \times H^1(\Lambda^1 TM) \longrightarrow H^2(\Lambda^1 TM)$$

(often called **the Schouten bracket**; see [Koz]). We have also a multiplication map

$$\cup : H^1(\Lambda^1 TM) \times H^1(\Lambda^1 TM) \longrightarrow H^2(\Lambda^2 TM)$$

Both of these maps have their counterparts acting on differential forms:

$$\begin{aligned} [\cdot, \cdot] &: \Lambda^{0,1}(\Lambda^1 TM) \times \Lambda^{0,1}(\Lambda^1 TM) \longrightarrow \Lambda^{0,2}(\Lambda^1 TM), \\ \cup &: \Lambda^{0,1}(\Lambda^1 TM) \times \Lambda^{0,1}(\Lambda^1 TM) \longrightarrow \Lambda^{0,2}(\Lambda^2 TM) \end{aligned} \quad (4.2.2)$$

Deformations of the complex structure operator $\bar{\partial}$ are parametrized by $\bar{\partial}_{new} = \bar{\partial} + A$, where A is a $(0, 1)$ -form with coefficients in TM . By definition,

$$\bar{\partial}_{new} f = \bar{\partial} f + \sum \lambda_i \cdot \bar{v}_i^\rightarrow(f), \quad (4.2.3)$$

where $A = \sum \lambda_i \bar{v}_i^\rightarrow$, $\lambda_i \in \Lambda^{0,1}(M)$, \bar{v}_i^\rightarrow a holomorphic vector field. Kodaira–Spencer theory for the deformations of complex structure can be boiled down to the following statement.

Theorem 4.2.1: (Kodaira–Spencer, Kuranishi) The operator $\bar{\partial}_{new}$ defines a complex structure if and only if

$$[A, A] = \bar{\partial} A, \quad (4.2.4)$$

where $\bar{\partial} : \Lambda^{0,q}(M, TM) \longrightarrow \Lambda^{0,q+1}(M, TM)$ is the $\bar{\partial}$ -operator on the forms with coefficients in TM . The forms $A_1, A_2 \in \Lambda^{0,1}(M, TM)$ define isomorphic complex structures if and only if $A_1 - A_2 = \bar{\partial} x$, for x a holomorphic vector field on M .

■

4.3 Tian–Todorov lemma.

Under the assumptions of Subsection 4.1, it is possible to relate the maps $[\cdot, \cdot]$ and \cup of (4.2.2).

Lemma 4.3.2: (Tian–Todorov) [Ti], [Tod1]. Let

$$\eta : \Lambda^{0,q}(\Lambda^i TM) \longrightarrow \Lambda^{0,q}(\Omega^{n-i} M)$$

be the map of (4.1.1). Then

$$\eta([\alpha, \beta]) = \partial(\eta(\alpha) \bullet_Y \eta(\beta)) - \partial\eta(\alpha) \bullet_Y \eta(\beta) - \eta(\alpha) \bullet_Y \partial\eta(\beta), \quad (4.3.5)$$

where $\partial : \Lambda^{0,q}(\Omega^p M) = \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M) = \Lambda^{0,q}(\Omega^{p+1} M)$ is the standard Dolbeault differential on (p, q) -forms, $\alpha, \beta \in \Lambda^{0,1}(M, TM)$, and

$$\bullet_Y : \Lambda^{p,q}(M) \times \Lambda^{p',q'}(M) \longrightarrow \Lambda^{p+p'-n, q+q'}(M)$$

is the so-called “Yukawa product” on forms, $a \bullet_Y b = \eta\left(\eta^{-1}(a) \cup \eta^{-1}(b)\right)$

Proof: Both sides of (4.3.5) contain differential operators of first order. Therefore it suffices to prove (4.3.5) for $M = \mathbb{C}^n$ with the flat metric. In this case (4.3.5) is proved trivially by computation. ■

4.4 Green operators and the Schouten bracket.

The Tian–Todorov lemma provides a way to solve the equation (4.2.4) explicitly, thus proving the Torelli theorem for Calabi–Yau manifolds (known as Bogomolov–Tian–Todorov theorem).

Let

$$\bullet_Y : \Lambda^{p,q}(M) \times \Lambda^{p',q'}(M) \longrightarrow \Lambda^{p+p'-n, q+q'}(M)$$

be the multiplication map obtained from

$$\cup : \Lambda^{0,q}(\Lambda^{n-p} TM) \times \Lambda^{0,q'}(\Lambda^{n-p'} TM) \longrightarrow \Lambda^{0,q+q'}(\Lambda^{2n-p-p'} TM)$$

by twisting with η . The map \bullet_Y is called **the Yukawa product**.

Applying η to both sides of (4.2.4), and then using the Tian–Todorov lemma, we obtain the equation

$$\partial(a \bullet_Y a) = \bar{\partial}a, \quad (4.4.6)$$

where $a = \eta(A)$ is a section of $\Lambda^{n-1,1}(M)$ satisfying $\partial(a) = 0$. This equation is easier to solve than (4.2.4).

Lemma 4.4.3: ($\partial\bar{\partial}$ -lemma) [GH] Let $\omega \in \Lambda^{p,q}(M)$ be a form which is $\bar{\partial}$ -closed and ∂ -exact. Then $\omega = \partial\bar{\partial}\theta$ for some $\theta \in \Lambda^{p-1,q-1}(M)$.

■

Let $G_{\bar{\partial}} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q-1}(M)$ be the Green operator which inverts the differential operator $\bar{\partial}$. This operator is defined by $G_{\bar{\partial}} = G_{\Delta} \circ \bar{\partial}^*$, where G_{Δ} is the usual Green operator inverting the Laplacian. Consider the standard orthogonal decomposition

$$\Lambda^{p,q} = \text{im } \bar{\partial} \oplus \text{im } \bar{\partial}^* \oplus \mathcal{H}^{p,q},$$

where $\mathcal{H}^{p,q}$ is the space of harmonic (p,q) -forms. Then $G_{\bar{\partial}}$ satisfies

$$\begin{aligned} G_{\bar{\partial}} \Big|_{\text{im } \bar{\partial}^*} &= 0, \\ G_{\bar{\partial}} \Big|_{\mathcal{H}^{p,q}} &= 0, \\ G_{\bar{\partial}} \circ \bar{\partial} \Big|_{\text{im } \bar{\partial}} &= Id \end{aligned} \quad (4.4.7)$$

By Kodaira, $G_{\bar{\partial}}$ commutes with ∂ :

$$G_{\bar{\partial}} \circ \partial = -\partial \circ G_{\bar{\partial}}.$$

The following lemma is a key statement in constructing Tian-Todorov coordinates.

Lemma 4.4.4: Let $a \in \mathcal{H}^{n-1,1}(M)$ be a harmonic form on M . We define a form $a_k \in \Lambda^{n-1,1}(M)$ recursively by

$$\begin{cases} a_k := \sum_{\substack{i+j=n-1 \\ i,j \geq 0}} G_{\bar{\partial}} \partial (a_i \bullet_Y a_j), k \geq 1 \\ a_0 := a, \end{cases}$$

where

$$\bullet_Y : \Lambda^{n-1,1}(M) \times \Lambda^{n-1,1}(M) \longrightarrow \Lambda^{n-2,2}(M)$$

is the Yukawa product.

Then

$$\bar{\partial} \left(\sum_{i+j=k-1} \partial(a_i \bullet_Y a_j) \right) = 0$$

for all $k > 0$.

Proof: For $\alpha, \beta \in \Lambda^{p,q}(M)$, let $[\alpha, \beta] := \partial(\alpha \bullet_Y \beta)$. This operation is Yukawa dual to the Schouten bracket of Subsection 4.2. Since the Schouten bracket commutes with the holomorphic structure operator, and the Yukawa duality operator $\eta : \Omega^i(M) \longrightarrow \Lambda^{n-1}TM$ is holomorphic, we have

$$\bar{\partial}[x, y] = [\bar{\partial}x, y] + [x, \bar{\partial}y]. \quad (4.4.8)$$

Therefore,

$$\bar{\partial}[a_i, a_j] = [\bar{\partial}a_i, a_j] + [a_i, \bar{\partial}a_j]$$

Using induction, we may assume that

$$\bar{\partial} \sum_{i+j=p-1} [a_i, a_j] = 0.$$

for $p < n$. Using the $\partial\bar{\partial}$ -lemma and (4.4.8), we obtain that for such p , the form

$$\sum_{i+j=p-1} [a_i, a_j] = \partial \sum_{i+j=p-1} a_i \bullet_Y a_j$$

is $\bar{\partial}$ -exact. Therefore, by (4.4.7), we have

$$\bar{\partial}a_p = \bar{\partial}G_{\bar{\partial}} \left(\sum_{i+j=p-1} [a_i, a_j] \right) = \sum_{i+j=p-1} [a_i, a_j]. \quad (4.4.9)$$

This implies that

$$\bar{\partial}[a_p, a_q] = [\bar{\partial}a_p, a_q] + [a_p, \bar{\partial}a_q] = \sum_{i+j=p-1} [[a_i, a_j], a_q] + \sum_{i+j=q-1} [a_p, [a_i, a_j]].$$

Therefore,

$$\bar{\partial} \sum_{p+q=k-1} [a_p, a_q] = \sum_{i+j+l=k-2} \left([[a_i, a_j], a_l] + [a_i, [a_j, a_l]] \right) \quad (4.4.10)$$

Clearly, the bracket $[\cdot, \cdot]$ is supersymmetric:

$$[\alpha, \beta] = (-1)^{(p-1)(q-1)} [\beta, \alpha], \quad \alpha \in \Lambda^{n-p,p}(M), \beta \in \Lambda^{n-q,q}(M)$$

This implies that the sum (4.4.10) is zero. Lemma 4.4.4 is proven. ■

4.5 Canonical coordinates.

We use the notation introduced in Subsection 4.4. For any $a \in \mathcal{H}^{n-1,1}(M)$, consider the sum $\sum a_i$, where the a_i are the forms defined in Lemma 4.4.4. Clearly from Hodge theory, the operator $G_{\bar{\partial}}$ is compact and ∂ is elliptic. From spectral theory it might be seen that the sum $\sum a_i$ converges absolutely for sufficiently small a . Let $A \in \Lambda^{0,1}(TM)$ be the $(0,1)$ -form with coefficients in TM which corresponds to $\sum a_i \in \Lambda^{0,1}(\Omega^{n-1}(M))$ by the Yukawa isomorphism (4.1.1).

Claim 4.5.5: The form A satisfies the equation (4.2.4) of Kodaira-Spencer.

Proof: By construction, $\partial a_i = 0$, for all i . Therefore, by Tian-Todorov, to prove (4.2.4) it suffices to show that $A' = \sum a_i$ satisfies $\partial(A' \bullet_Y A') = \bar{\partial} A'$. By definition,

$$\bar{\partial} A' = \sum_p \bar{\partial} G_{\bar{\partial}} \left(\sum_{i+j=p-1} [a_i, a_j] \right).$$

By (4.4.9) and Lemma 4.4.4, we have

$$\bar{\partial} G_{\bar{\partial}} \sum_{i+j=p-1} [a_i, a_j] = \sum_{i+j=p-1} [a_i, a_j].$$

Therefore,

$$\bar{\partial} A' = \sum_p \sum_{i+j=p-1} [a_i, a_j] = [A', A'].$$

■

We obtain that A defines a complex structure on M , in the sense of Kuranishi (Theorem 4.2.1). We have proved the following theorem.

Theorem 4.5.6: (Bogomolov – Tian – Todorov) [B1] [Ti], [Tod1] For each cohomology class $\alpha \in H^1(M, TM)$, consider the corresponding harmonic form $a \in \mathcal{H}^{n-1,1}(M)$. Let $B \subset H^1(M, T)$ be an open ball where the series $\sum a_i$ converges. Let X be the moduli space of M . Let $\varphi : B \rightarrow X$ be the map associating to α the complex structure defined by $A = \sum a_i$. Then, for B sufficiently small, the map φ is an open embedding, which is independent of the choice of Kähler metric on M .

Proof: Immediately follows from Theorem 4.2.1. ■

Definition 4.5.7: Let M be a Calabi-Yau manifold and $x \in X$ be a point in its moduli space. Consider the coordinates in a neighbourhood of X , defined as in Theorem 4.5.6. These coordinates are called **Tian-Todorov coordinates on X** .

Remark 4.5.8: Tian–Todorov coordinates depend on the base point $x \in X$. The translation between these coordinates, for different base points, is not necessarily flat.

Remark 4.5.9: In the context of the Mirror Symmetry, Tian–Todorov coordinates were introduced in [BCOV]. Since then, these coordinates have been common in the physics literature. The definition above comes from A. Todorov ([Tod1]; also [Ti]).

5 The Mirror Conjecture.

5.1 Mirror Symmetry from a mathematician’s point of view.

Definition 5.1: Let M be a Calabi-Yau manifold, and X be its deformation space. The Tian-Todorov lemma provides that for every $I \in X$, there are canonical flat¹ coordinates in a neighbourhood U_I of X (first introduced in [BCOV]; for a formal definition, see Section 4 of the present paper). These coordinates are called **Tian-Todorov coordinates on X** . Denote by K the standard line bundle over X , $K|_x = H_x^{n,0}(M)$. Let \mathcal{A} be a VFA over U , where U is an open subset in a linear space L . We say that **Yukawa VFA is equivalent to \mathcal{A}** if for all $I \in X$ there exist

¹These coordinates depend on I , in such a way that translation between U_I for different I is *not* flat.

- (i) An open set $U_0 \subset U$ and an isomorphism $\varphi_I : U_0 \longrightarrow U_I$, preserving flat coordinates, where U_I is a Tian-Todorov neighbourhood of I equipped with flat coordinates.
- (ii) A trivialization θ of K over $U_I \subset \text{Comp}$.

Let Yu be the Yukawa VFA on U_I defined by θ , and \mathcal{H} be its total space, $\mathcal{H} = H^*(M) \times U_I$.

- (iii) An isomorphism $\varphi_I^* A \sim \mathcal{H}$ of vector bundles over U_I inducing an isomorphism of variations of Frobenius algebras.²

Mirror Conjecture: Let M be a Calabi-Yau manifold. We say that **the Mirror Conjecture holds for M** if there exist a Calabi-Yau manifold W (called **Mirror dual** Calabi-Yau manifold) such that the power series (3.1) converges for M and W in nonempty open sets, the assumptions of Proposition 3.4 are satisfied for M and W , Yukawa VFA for M is equivalent to Quantum VFA for W , and Yukawa VFA for W is equivalent to Quantum VFA for M .

Remark 5.2: The quantum VFA $Q = \oplus H^{p,q}(M)$ has a subalgebra $Q^{p,p} = \oplus H^{p,p}$. Similarly, the Yukawa VFA has a subalgebra

$$Y^{n-p,p} = \oplus H_I^{n-p,p}(M).$$

Often, Mirror Symmetry is understood as an isomorphism between these subalgebras. Our proof of Mirror Symmetry for holomorphically symplectic manifolds works equally well in both of these cases.

Remark 5.3: It is important to notice that, in our definition, the Quantum VFA depends on the trivialization θ of the linear bundle K . To wit, Mirror Symmetry gives information about multiplication in the cohomology (by identifying Yukawa and Quantum rings) but only up to a scalar multiple.

The following theorem is the main result of this paper. It is proven in Section 12.

²The VFA structure on $\varphi_I^* A$ is the pullback of the VFA on $A|_{U_0}$.

Theorem 5.4: Let M be a compact holomorphically symplectic manifold, which is generic in its deformation class.³ Then the Mirror Conjecture holds for M , which is Mirror dual to itself.

5.2 Appendix: Mirror Symmetry from a physicist's point of view.

For a physicist, it is more natural to think of Mirror Symmetry in terms of the so-called correlation functions. For completeness, we give an (equivalent to ours) statement of the Mirror Conjecture in these terms. We assume that M and W are mirror dual and explain what it means in the language of correlators.

Let

$$x \in H^{1,1}(W), \quad y \in \text{Comp}(M), \quad y = \varphi_I(x),$$

where φ_I is the Tian-Todorov map. Then Yukawa (B-model) correlations are i -forms on $\oplus H_y^{p,q}(M)$ defined by

$$\langle \alpha_1 \dots \alpha_i \rangle \longrightarrow \int_M \alpha_1 \bullet_Y \dots \bullet_Y \alpha_i \quad (5.1)$$

where \bullet_Y is the Yukawa product associated with the complex structure y . The quantum (A-model) correlations are, similarly, i -forms on $\oplus H^{p,q}(W)$ defined by

$$\langle \alpha_1 \dots \alpha_i \rangle \longrightarrow (\alpha_1 \bullet_Q \dots \bullet_Q \alpha_{i-1}, \alpha_i) \quad (5.2)$$

where \bullet_Q is the quantum product associated with $x \in H^{1,1}(W)$ and (\cdot, \cdot) is the Poincaré form on the quantum cohomology (which is a part of the Frobenius structure on quantum cohomology). The **marginal correlators** are, in the B-model situation, n -forms on $T_y \text{Comp} = H_y^{n-1,1}(M)$ defined by (5.1), and, on the A-model side, n -forms on $H^{1,1}(W)$ defined by (5.2). The “parameter space” for the A-model is the convergence domain of the Dubrovin potential in $H^{1,1}(W)$ (or a quotient thereof by an action of a discrete group, but we are interested only in the local structure of the parameter space). The “parameter space” for the A-model is the image of φ_I in $\text{Comp}(M)$, or, what is equivalent, an open subset in $H^{n-1,1}(M)$. Denote the parameter spaces of the respective models by P_A, P_B . The Mirror Conjecture says that (shrinking P_A, P_B if necessary) under the appropriate flat

³ $\text{Pic}(M) = 0$ will suffice.

identification of $P_A \subset H^{1,1}(W)$, $P_B \subset H^{n-1,1}(W)$, $H^{1,1}(M) \cong H^{n-1,1}(M)$, the bundles $\oplus H_y^{p,q}(M)$, $\oplus H^{p,q}(W)$ can be identified in such a way that correlation functions are mapped to correlation functions (this corresponds to compatibility of φ_I with the Frobenius structure), and marginal correlators to marginal correlators (this corresponds to compatibility of φ_I with the embeddings $\tau : TU \longrightarrow \mathcal{A}^{gr}$). Notice that, for a physicist, only weak VFA are of significance. Still, it is easier and conceptually more natural to formulate the Mirror Conjecture using the VFA and weak \mathbb{C} -VHS.

6 Quantum VFA for holomorphically symplectic manifolds.

Let M be a compact holomorphically symplectic manifold which is generic in its deformation class. In [V-Sym] it is proven that all closed complex subvarieties of M are even-dimensional. In particular, M has no curves (and no integral (1,1)-cycles). Therefore, all non-trivial Gromov-Witten classes for M vanish. Applying the formula (3.1), we find that the quantum deformation potential is

$$\Phi_\omega(\gamma) = \varepsilon(\exp(x)),$$

where $\varepsilon : H^*(M) \longrightarrow \mathbb{C}$ is the trace form projecting $H^*(M)$ to $H^{2n}(M) = \mathbb{C}$, and $\exp : H^*(M) \longrightarrow H^*(M)$ maps x to $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$. Note that $\Phi_\omega(\gamma)$ is independent of ω .

For an arbitrary supercommutative Frobenius algebra A over \mathbb{C} or \mathbb{R} , the function $\Phi(x) = \varepsilon(\exp(x))$ satisfies the conditions of Definition 3.1. We call it **the primary Dubrovin potential associated with the algebra A** .

Proposition 6.1: Let A be a supercommutative Frobenius algebra over a field of char 0, and Φ be a primary Dubrovin potential. Then

$$\left. \frac{\partial^3 \Phi}{\partial_a \partial_b \partial_c} (x, y, z) \right|_t = \varepsilon(xyz \exp(t)),$$

where a, b, c are arbitrary vectors in $T_t S = A$.

Proof: Clear. ■

Corollary 6.2: Let $C_{a,b}^c(x) : A \times A \longrightarrow A$ be the product in $T_x S$ determined by the tensor $\left. \frac{\partial^3 \Phi}{\partial_a \partial_b \partial_c} \right|_x$ as in Proposition 3.2. Let $C_{a,b}^c(0)$ be the original product in A . Then

$$C_{a,b}^c(t)(x, y) = C_{a,b}^c(0)(x, \exp(t)y).$$

Proof: Follows from Proposition 6.1. ■

We obtain the following description of Quantum VFA.

Definition 6.3: Let $A = A_0 \oplus A_1 \oplus \dots \oplus A_n$ be a graded Frobenius algebra and $U \subset \text{Aff}(A_2)$ be a submanifold of $\text{Aff}(A_2)$. Let $\mathcal{A} = U \times A$ be the trivial bundle over U , with fiber A , and ∇_0 be the trivial connection in \mathcal{A} . There is a natural embedding $TU \xrightarrow{\tau} \mathcal{A}$ and a natural multiplication $\cdot : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$. Let $C_{a,b}^c : \mathcal{A} \otimes TU \longrightarrow \mathcal{A}$ be the map associating $\tau(t) \cdot a$ to $a \otimes t$, where $a \in \mathcal{A}$, $t \in TU$, and $t \cdot a$ is multiplication in \mathcal{A} . Consider $C_{a,b}^c$ as a map from \mathcal{A} to $\mathcal{A} \otimes \Lambda^1 U$. Let ∇_m be the connection in \mathcal{A} defined by $\nabla_m = \nabla_0 + C_{a,b}^c$. Consider the decomposition of \mathcal{A} onto even and odd parts, $\mathcal{A} = \mathcal{A}^{\text{odd}} \oplus \mathcal{A}^{\text{even}}$, with Hodge filtration in \mathcal{A}^{odd} , $\mathcal{A}^{\text{even}}$ defined as in Proposition 3.4, $\mathcal{A}_k^{\text{even}} = \bigoplus_{i=0}^k \mathcal{A}_{2i}$, $\mathcal{A}_k^{\text{odd}} = \bigoplus_{i=0}^k \mathcal{A}_{2i+1}$. Since the filtration $\mathcal{A}_0^{\text{even}} \subset \mathcal{A}_1^{\text{even}} \subset \dots$, $\mathcal{A}_0^{\text{odd}} \subset \mathcal{A}_1^{\text{odd}} \subset \dots$ comes from the grading, we may naturally identify \mathcal{A} with its associated graded bundle \mathcal{A}^{gr} . Clearly,

$$\left(\mathcal{A} = \mathcal{A}^{\text{odd}} \oplus \mathcal{A}^{\text{even}}, \nabla_m, \mathcal{A}_0^{\text{even}} \subset \mathcal{A}_1^{\text{even}} \subset \dots, \mathcal{A}_0^{\text{odd}} \subset \mathcal{A}_1^{\text{odd}} \subset \dots \right. \\ \left. \cdot : \mathcal{A}^{gr} \times \mathcal{A}^{gr} \longrightarrow \mathcal{A}^{gr}, \right)$$

gives a VFA structure on \mathcal{A} . Such VFA is called **the trivial VFA associated with A , $U \subset \text{Aff}(A_2)$** .

Theorem 6.4: Let M be a Kähler manifold without holomorphic curves, $A = H^*(M)$, $U = \text{Aff}(H^{1,1}(M))$, and \mathcal{A} be the trivial bundle over U with the fiber $H^*(M)$, equipped with a Quantum VFA structure. Then \mathcal{A} is the trivial VFA over U .

Proof: Clear. ■

This finishes the calculation of quantum cohomology VFA in the case of a holomorphically symplectic manifold which is generic in its deformation class.

Appendix. To see how our computation works in physicists' language, the reader can check the following trivial claim. The “marginal correlations” of Section 5 are easy to write down. Let $S^{1,1}$ be the base for the Quantum VFA, denoted by \mathcal{A} , and $i : TS^{1,1} \hookrightarrow B$ be the homomorphism which is a part of the VFA structure. Since the bundle $TS^{1,1}$ is trivial, we may consider i as a map from the bundle $TS^{1,1} = H^{1,1}(M) \times S^{1,1}$ to \mathcal{A} . By definition, marginal correlations are n -linear functions on $H^{1,1}(M)$, depending on a base point $s \in S^{1,1}$. From Theorem 6.4 it follows that the marginal correlation function is a map from $S^{1,1}$ to the space $S^n(H^{1,1}(M))^*$ of symmetric n -forms on $H^{1,1}(M)$, given by

$$\langle \alpha_1, \dots, \alpha_n \rangle_x = \left(i(\alpha_1) \bullet_x \dots \bullet_x i(\alpha_{n-1}), i(\alpha_n) \right)$$

as in (5.2).

Claim 6.5: Let \mathcal{A} be the Quantum VFA associated with a manifold which has no rational curves. Then the “marginal correlator” map

$$\langle \cdot, \cdot, \dots \cdot \rangle_x : S^{1,1} \longrightarrow S^n(H^{1,1}(M))^*$$

is constant.

Proof: Clear. ■

7 Periods of holomorphically symplectic manifolds and Yukawa VFA.

For clarifications and missing definitions, the reader is referred to [V].

In this section, we give an outline of how the Yukawa product on the cohomology of a holomorphically symplectic manifold M can be expressed through a group action on its period space.

Let M be a compact holomorphically symplectic manifold of Kähler type, and $Comp$ be its coarse marked deformation space. Assume¹ that $\dim H^{2,0}(M) = 1$. In [Beau] (Remarques, p. 775), Beauville defined a canonical non-degenerate symmetric 2-form $(\cdot, \cdot)_{\mathcal{H}}$ on $H^2(M)$ of signature $(n-3, 3)$, $n = \dim H^2(M)$. We call this form **the Bogomolov-Beauville**

¹This assumption is needed only to simplify the exposition. Details in [V], Section 3.

pairing. For a complex structure $I \in \text{Comp}$, let $\rho_I : \mathfrak{u}(1) \longrightarrow \text{End}(H^*(M))$ be the linear operator mapping $t \in \mathfrak{u}(1) \equiv \mathbb{R}$ to the endomorphism

$$\omega^{p,q} \longrightarrow (p - q)\sqrt{-1} \omega^{p,q}t,$$

for all $\omega^{p,q} \in H_I^{p,q}(M)$. Let $\mathfrak{g}_0(M) \subset \text{End}(H^*(M))$ be the Lie algebra generated by ρ_I for all $I \in \text{Comp}$, and $G_0(M) \subset \text{End}(H^*(M))$ be the corresponding Lie group. As [V], Theorem 5.1 implies, ρ_I preserves the form $(\cdot, \cdot)_{\mathcal{H}}$, for all $I \in \text{Comp}$. This defines a Lie algebra homomorphism

$$\rho : \mathfrak{g}_0(M) \longrightarrow \mathfrak{so}\left(H^2(M), (\cdot, \cdot)_{\mathcal{H}}\right). \quad (7.1)$$

Theorem 7.1: ([V], Theorem 12.2) The map (7.1) is an isomorphism.

■

Let u be the unit vector in $\mathfrak{u}(1)$, and $ad I = \rho_I(u) : H^2(M) \longrightarrow H^2(M)$ be the corresponding endomorphism. Let $X \subset \mathfrak{g}_0(M)$ be the $G_0(M)$ -orbit of $ad I$, for some I . As the following claim implies, X is independent of $I \in \text{Comp}$.

Claim 7.2: For all $L \in \text{Comp}$, the endomorphism $ad L \in \text{End}(H^*(M))$ belongs to X .

Proof: By definition, the endomorphism $ad L$ belongs to $\mathfrak{g}_0(M)$. On the other hand, we proved that $\mathfrak{g}_0(M)$ is naturally isomorphic to $\mathfrak{so}(\mathbb{V})$, where \mathbb{V} is the space $H^2(M, \mathbb{R})$ equipped with the Bogomolov-Beauville form $(\cdot, \cdot)_{\mathcal{H}}$. Consider $ad L$ as an endomorphism of \mathbb{V} . The operator $ad L$ has two non-zero eigenvalues, $2\sqrt{-1}$ and $-2\sqrt{-1}$, corresponding to $H_L^{2,0}(M)$ and $H_L^{0,2}(M)$ respectively. Let $V_{\neq 0}$ be the subspace of V corresponding to these eigenvalues,

$$V_{\neq 0} = \left(H_L^{2,0}(M) \oplus H_L^{0,2}(M) \right) \cap H^2(M, \mathbb{R}).$$

Since $h^{2,0}(M) = 1$, the space $V_{\neq 0}$ is 2-dimensional. Then $(\cdot, \cdot)_{\mathcal{H}}$ restricted to $V_{\neq 0}$ is positive definite. Now, Claim 7.2 is an implication of the following linear algebra observation.

- For all $x \in \mathfrak{so}(\mathbb{V})$ with two non-zero eigenvalues $2\sqrt{-1}$ and $-2\sqrt{-1}$ of multiplicity one, and such that $(\cdot, \cdot)_{\mathcal{H}}$ restricted to $V_{\neq 0}$ is positive definite, we have $x \in X$.

■

Denote the map $I \longrightarrow ad I$ by $P_o : Comp \longrightarrow X$. Identifying $\mathfrak{g}_0(M)$ with its dual space, we can consider X as a coadjoint orbit. Therefore, X is equipped with a natural complex structure (as a coadjoint orbit of a reductive group).²

For $I \in Comp$, let $G_0^I(M) \subset G_0(M)$ be the stabilizer of $P_o(I) \in \mathfrak{g}_0(M)$, under the adjoint action of $G_0(M)$ on $\mathfrak{g}_0(M)$:

$$G_0^I(M) = \{g \in G_0(M) \mid g(\rho_I(u)) = ad I\}.$$

Let $Comp$ be the moduli space of M , and \underline{K} be the standard holomorphic line bundle over $Comp$ with fibers $\underline{K}|_I = H_I^0(\Omega^n(M))$, $n = \dim_{\mathbb{C}} M$, $I \in Comp$. Let $\underline{\Omega}$ be the standard holomorphic line bundle over $Comp$ with fibers $\underline{\Omega}|_I = H_I^{2,0}(M)$. By definition, $\underline{K} = \underline{\Omega}^{n/2}$. Let $\Omega \subset X \times H^2(M, \mathbb{C})$ be the line bundle over X , with fibers

$$\Omega|_x = \{l \in H^2(M, \mathbb{C}) \mid xl = 2\sqrt{-1}l\}.$$

Clearly, $\underline{\Omega} = P_o^* \Omega$. Therefore, $P_o^*(\Omega)^{\frac{n}{2}} = \underline{K}$. The bundle Ω is naturally $G_0(M)$ -equivariant. This gives an action of $G_0^I(M)$ on $\Omega|_{ad I}$, for all $I \in Comp$. Since the fiber of $\Omega^{n/2}$ in $ad I \in X$ is naturally isomorphic to the fiber of \underline{K} in $I \in Comp$, we obtain also a natural action of $G_0^I(M)$ on $\underline{K}|_I \cong H_I^{n,0}(M)$.

The following theorem will be proven in Section 10.

Theorem 7.3: *For $I \in Comp$, consider the natural action of the group $G_0^I(M)$ on $H^*(M)$, $H_I^{n,0}(M)$. Let*

$$Y_I : H^*(M) \times H^*(M) \longrightarrow H^*(M) \otimes H_I^{n,0}(M)$$

be the Yukawa product. Then Y_I commutes with the action of $G_0^I(M)$.

Consider the trivial bundle B over X , with the fiber $H^*(M)$, $B = X \times H^*(M)$. The action of $G_0(M)$ on $H^*(M)$, X defines a $G_0(M)$ -equivariant structure on B . Theorem 7.3 automatically allows us to define an equivariant

² By Bogomolov, the map P_o is an open covering, and is holomorphic with respect to this complex structure.

multiplicative structure $\bullet_B : B \times B \longrightarrow B \otimes K$ on B which can be, a priori, dependent on the choice of I . For $x \in X$, $h_1, h_2 \in B|_x$, $x = g(P_o(I))$, $\lambda \in \underline{K}|_x$, we set

$$h_1 \bullet_{B,g} h_2|_x = g(\lambda)g(Y_I(g^{-1}h_1, g^{-1}h_2))|_{g(x)}. \quad (7.2)$$

To prove that (7.2) defines an equivariant product $h_1 \bullet_B h_2$ on B , we have to show that this $\bullet_{B,g}$ is independent on the choice of g . For two possible choices g, g' , we have $x = g^{-1}g' \in G_0^I(M)$, because $G_0^I(M)$ is the stabilizer of $P_o(I)$. Clearly, from the definition,

$$h_1 \bullet_{B,g} h_2 = x^{-1} (x(h_1) \bullet_{B,g'} x(h_2)).$$

Theorem 7.3 implies that $h_1 \bullet_{B,g} h_2 = h_1 \bullet_{B,g'} h_2$.

Theorem 7.4: *Let $P_o^*(B)$ be a trivial bundle over $Comp$ with fiber $H^*(M)$, and*

$$\bullet_Y : P_o^*(B) \times P_o^*(B) \longrightarrow P_o^*(B) \otimes K$$

be the Yukawa product in $P_o^(B)$. Let \bullet_B be the equivariant product in B associated to $I \in Comp$ as above. Then \bullet_B is independent of I , and \bullet_Y coincides with the pullback of \bullet_B .*

Theorem 7.4 will be proven in Section 10.

8 The $Spin(5)$ -action on $H^*(M)$.

We refer to [V] for details of definitions and for missing proofs. A hyperkähler manifold is a Riemannian manifold M equipped with three complex structures I, J and K , such that $I \circ J = -J \circ I = K$ and M is Kähler with respect to I, J and K . Relations between I, J and K imply that there is an action of the quaternions in its tangent space.

Let M be a complex manifold which admits a hyperkähler structure. A simple linear algebra argument implies that M , considered as a complex manifold with the complex structure coming from the standard embedding

$\mathbb{C} \hookrightarrow \mathbb{H}$, is equipped with a holomorphic symplectic form.¹ The Calabi-Yau theorem [Y] shows that, conversely, every compact holomorphically symplectic Kähler manifold admits a hyperkähler structure, which is uniquely defined by these data.

8.1 Quaternionic-Hermitian spaces and differential forms over a hyperkähler manifold.

Let M be a holomorphically symplectic Kähler manifold, and

$$\mathcal{H} = (I, J, K, (\cdot, \cdot))$$

be a hyperkähler structure on M , which exists and is unique by the Calabi-Yau theorem. Consider the Kähler classes $\omega_I, \omega_J, \omega_K \in H^2(M, \mathbb{R})$ associated with I, J, K , and let $L_{\omega_\alpha}, \Lambda_{\omega_\alpha} : H^*(M) \rightarrow H^*(M)$, $\alpha = I, J, K$, be the corresponding Hodge operators.

Theorem 8.1: ([V-so], Theorem 1) The operators $L_{\omega_\alpha}, \Lambda_{\omega_\alpha}$, $\alpha = I, J, K$, generate a 10-dimensional Lie algebra $\mathfrak{g}(\mathcal{H})$, which is naturally isomorphic to $\mathfrak{so}(4, 1)$.

Sketch of a proof: Consider the Hodge operators $L_{\omega_\alpha}, \Lambda_{\omega_\alpha}$, $\alpha = I, J, K$, acting on differential forms over M . These operators commute with the Laplacian (by Kodaira) and hence act on cohomology. Therefore, to prove Theorem 8.1 it suffices to show that $L_{\omega_\alpha}, \Lambda_{\omega_\alpha}$ generate a Lie algebra $\mathfrak{so}(4, 1)$ acting on differential forms.

Let T be a vector space equipped with a quaternionic action and a positive definite \mathbb{R} -valued symmetric pairing (\cdot, \cdot) such that the quaternions I, J, K are orthogonal operators with respect to (\cdot, \cdot) . Such a vector space is called **quaternionic-Hermitian**. For every quaternionic-Hermitian vector space T , we can consider an action of the Lie algebra $\mathfrak{g}(T)$ on $\Lambda_{\mathbb{R}}^*(T)$, defined in the same way as for differential forms: the operators L_{ω_α} act as exterior multiplication by ω_α , and the operators Λ_{ω_α} are adjoint to L_{ω_α} with respect to the positive definite metric induced by (\cdot, \cdot) .

Let $x \in M$, and $T_x M$ be the tangent space at x . Since M is hyperkähler, the space $T_x M$ is equipped with a natural quaternionic-Hermitian structure. The action of $\mathfrak{g}(\mathcal{H})$ on $\Lambda^*(M)$ comes from an action of $\mathfrak{g}(\mathcal{H})$ on

¹Precisely, let (I, J, K) be the standard generators of the quaternion algebra, and ω_α , $\alpha = I, J, K$ Kähler forms associated with corresponding the complex structures. Then $\omega_J + \sqrt{-1}\omega_K$ is holomorphically symplectic with respect to I .

$\Lambda^*(T_x M)$. Therefore, to prove Theorem 8.1 it suffices to show that for every quaternionic-Hermitian space T , the Lie algebra $\mathfrak{g}(T)$ is isomorphic to $\mathfrak{so}(1, 4)$.

Let $T = \oplus T_i$ be an orthogonal decomposition of T onto a direct sum of \mathbb{H} -invariant subspaces. Clearly, $\Lambda^*(T) = \bigotimes_i \Lambda^* T_i$, and the action of $\mathfrak{g}(\mathcal{H})$ on $\bigotimes \Lambda^* T_i$ is multiplicative with respect to this decomposition:

$$\begin{aligned} \forall t \in \Lambda^*(T), \quad t = \bigotimes t_i, \quad t_i \in \Lambda^*(T_i), \quad h \in \mathfrak{g}(\mathcal{H}) \\ h(t) = \sum_i t_1 \otimes \dots \otimes h(t_i) \otimes \dots \otimes t_n. \end{aligned} \quad (8.1)$$

Therefore, it suffices to show that $\mathfrak{g}(T)$ is canonically isomorphic to $\mathfrak{so}(1, 4)$ for $\dim_{\mathbb{H}} T = 1$. This is done by a computation. ² ■

Corollary 8.2: Consider the representation $\text{Spin}(4, 1) \longrightarrow \text{End}(H^*(M))$ corresponding to the action of $\mathfrak{g}(\mathcal{H}) \cong \mathfrak{so}(4, 1)$ on $H^*(M)$. Let $Z \cong \mathbb{Z}/2\mathbb{Z}$ be the center of $\text{Spin}(1, 4)$, and ι be the non-trivial element of Z . Then $\iota(\omega) = (-1)^{\dim \omega} \omega$. In particular, Z acts trivially on $H^{\text{even}}(M)$.

Proof: Using (8.1) as in the proof of Theorem 8.1, we reduce Corollary 8.2 to the case of action of $\text{Spin}(1, 4)$ on $\Lambda_{\mathbb{R}}^*(T)$, where T is a quaternionic-Hermitian space, $\dim_{\mathbb{H}} T = 1$. Let $G(T) \subset \Lambda_{\mathbb{R}}^*(T)$ be the Lie group corresponding to $\mathfrak{g}(T)$. Consider an action of $G(T)$ on the space $W = \Lambda_{\mathbb{R}}^{\text{odd}}(T)$ for $\dim_{\mathbb{H}} T = 1$. Let (\cdot, \cdot) be the bilinear symmetric form on W of signature $(4, 4)$,

$$a, b \longrightarrow (-1)^{\frac{\dim a + 1}{2}} a \cup b.$$

Since $\Lambda_{\mathbb{R}}^3(T) \cong \Lambda_{\mathbb{R}}^1(T)^*$ and T is quaternionic, W is equipped with a natural quaternionic structure. It is easy to check that $G(T)$ preserves (\cdot, \cdot) and is generated by quaternionic matrices (see e. g. [V-so], Appendix). This defines a non-trivial map $G(T) \longrightarrow \text{Sp}(W) \cong \text{Sp}(1, 1)$. The groups $G(T)$, $\text{Sp}(1, 1)$ are simple and have the same dimension. Therefore, the natural map $G(T) \longrightarrow \text{Sp}(W)$ is an isomorphism. This gives an explicit way to identify $G(T)$ and $\text{Sp}(1, 1)$.

We obtained that the Lie algebra $\mathfrak{g}(T) \cong \mathfrak{sp}(1, 1)$ acts on $\Lambda_{\mathbb{R}}^{\text{odd}}(\mathbb{H})$ as on its fundamental representation. Since $\text{Sp}(1, 1) \cong \text{Spin}(1, 4)$, and the central

²In [V-so], generators and relations of $\mathfrak{g}(\mathcal{H})$ were written down, then a root system found. This root system turns out to be B_2 , which gives an isomorphism $\mathfrak{g}(\mathcal{H}) \otimes \mathbb{C} \cong \mathfrak{so}(5, \mathbb{C})$. To find out what real form of $\mathfrak{so}(5, \mathbb{C})$ corresponds to $\mathfrak{g}(\mathcal{H})$, we constructed a non-zero homomorphism $\mathfrak{g}(\mathcal{H}) \longrightarrow \mathfrak{sp}(1, 1) \cong \mathfrak{so}(1, 4)$ (see also the proof of Corollary 8.2).

element of $\mathrm{Sp}(1, 1)$ acts on its fundamental representation by -1 , we obtain that $\iota \Big|_{\Lambda_{\mathbb{R}}^{\mathrm{odd}}(\mathbb{H})} = -1$. Similarly one checks that ι acts trivially on $\Lambda_{\mathbb{R}}^{\mathrm{even}}(\mathbb{H})$.
 ■

8.2 Appendix. Generators and relations for the Lie algebra $\mathfrak{so}(1, 4)$.

For the benefit of the reader, we give the relations in $\mathfrak{g}(T)$, computed in [V-so]. We abbreviate $L_{\omega_I}, L_{\omega_J}, L_{\omega_K}, \Lambda_{\omega_I}, \Lambda_{\omega_J}, \Lambda_{\omega_K}$ by $L_1, L_2, L_3, \Lambda_1, \Lambda_2, \Lambda_3$. Let $K_{ij} := [L_i, \Lambda_j]$, $i \neq j$, and $H : \Lambda^*(T) \longrightarrow \Lambda^*(T)$ act on $\Lambda^i(T)$ as multiplication by the scalar $\dim_{\mathbb{R}} T - 2i$. Then the following relations are true and define the Lie algebra $\mathfrak{g}(T)$ in a unique way:

$$\begin{aligned}
 [L_i, L_j] &= [\Lambda_i, \Lambda_j] = 0; \\
 [L_i, \Lambda_i] &= H; \quad [H, L_i] = 2L_i; \quad [H, \Lambda_i] = -2\Lambda_i \\
 K_{ij} &= -K_{ji}, \quad [K_{ij}, K_{jk}] = 2K_{ik}, \quad [K_{ij}, H] = 0 \\
 [K_{ij}L_j] &= 2L_i; \quad [K_{ij}\Lambda_j] = 2\Lambda_i \\
 [K_{ij}, L_k] &= [K_{ij}, \Lambda_k] = 0 \quad (k \neq i, j)
 \end{aligned} \tag{8.2}$$

9 Operator of Serre duality.

Let M be a holomorphically symplectic manifold of Kähler type, $\dim_{\mathbb{C}} M = n$. The canonical class of M is naturally trivialized by the form $\Omega^{n/2}$, where Ω is the holomorphic symplectic form. Let $\Omega^i(M) \cong (\Omega^{n-i}M)^*$ be the duality coming from this trivialization of the canonical class, and $\Omega^i(M) \cong (\Omega^i(M))^*$ be the duality defined by the holomorphically symplectic form. Combining these two maps, we obtain an isomorphism $\Omega^i(M) \cong \Omega^{n-i}M$. The corresponding map $\eta^{i,j} : H^i(\Omega^j(M)) \longrightarrow H^i(\Omega^{n-j}(M))$ of cohomology is called **the Serre duality homomorphism**. Let $\eta = \bigoplus_{i,j} \eta^{i,j}$. Clearly, η is an invertible endomorphism of the space $H^*(M)$.¹ Yukawa multiplication \bullet_Y can be expressed via η and the usual multiplication \cup in cohomology as

$$h_1 \bullet_Y h_2 = \eta^{-1} \left(\eta(h_1) \cup \eta(h_2) \right). \tag{9.1}$$

¹The operator η is an involution, as follows from the argument one uses to show that the standard Hodge operator $*$ is an involution.

Remark: The multiplication \bullet_Y is determined by the complex structure and the section of the canonical class, and the operator η depends on the choice of holomorphic symplectic form.

Let $\mathcal{H} = (I, J, K, (\cdot, \cdot))$ be a hyperkähler structure on M , and $G(\mathcal{H}, \mathbb{C}) \subset \text{End}(H^*(M, \mathbb{C}))$ be the Lie group corresponding to the Lie algebra $\mathfrak{g}(\mathcal{H}) \otimes \mathbb{C} = \mathfrak{so}(1, 4)$ of Theorem 8.1.

Theorem 9.1: Consider the Serre duality operator η acting on $H^*(M)$. Then η belongs to $G(\mathcal{H}, \mathbb{C})$.

The proof of Theorem 9.1 takes the rest of this section.

It is possible to describe explicitly the element of $G(\mathcal{H}, \mathbb{C})$ which corresponds to η . Let T be a quaternionic-Hermitian space, and $\mathfrak{g}(T) = \mathfrak{so}(1, 4)$ be the Lie algebra defined in the proof of Theorem 8.1. Let $\Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C} = \oplus \Lambda_I^{p,q}(T)$ be the Hodge decomposition of $\Lambda_{\mathbb{R}}^*(T)$ taken with respect to the complex structure I , and $ad I, H$ be endomorphisms of $\Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C}$, defined on $\Lambda_I^{p,q}(T) \subset \Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C}$ by

$$\omega^{p,q} \xrightarrow{ad I} (p - q)\sqrt{-1} \omega^{pq},$$

$$\omega^{p,q} \xrightarrow{H} (p + q - 2n)\omega^{pq},$$

where $n = \dim_{\mathbb{H}} T$.

Lemma 9.2: [V-so] The endomorphisms $ad I, H$ belong to $\mathfrak{g}(T)$. These operators generate a Cartan subalgebra² \mathfrak{h} for $\mathfrak{g}(T)$. The root system of \mathfrak{h} is

$$\pm H, \quad \pm\sqrt{-1} ad I, \quad \pm H \pm \sqrt{-1} ad I. \quad (9.2)$$

■

In the above situation, let w be the Weyl group element which maps

²Which is 2-dimensional, because $\mathfrak{g}(T) \cong \mathfrak{so}(1, 4)$ with the root system B_2 .

$$\begin{aligned}
& H - \sqrt{-1} \operatorname{ad} I \quad \text{to} \quad -H + \sqrt{-1} \operatorname{ad} I \\
& \text{and} \\
& H + \sqrt{-1} \operatorname{ad} I \quad \text{to} \quad H + \sqrt{-1} \operatorname{ad} I.
\end{aligned} \tag{9.3}$$

This element defines an automorphism Θ of the root system, and therefore an automorphism of $\mathfrak{g}(T)$. Since Dynkin diagram of $\mathfrak{g}(T)$ is B_2 , all automorphisms of $\mathfrak{g}(T)$ are inner, by well-known computation of $\operatorname{Out}(G)$ for classical groups (see, for instance, [VO]). Therefore, we may consider Θ as an element of $G(T)/Z$, where $G(T) \subset \operatorname{End}(T)$ is the Lie group corresponding to $\mathfrak{g}(T)$ and Z is centre of $G(T)$. Further on, we consider Θ as an element in $G(T)$ acting on $\mathfrak{g}(T)$ as we just described. Since $Z = (\pm 1)$ (also [VO]), this involves a choice of one of two possible $\Theta \in G(T)$. We specify this choice shortly thereafter.

We describe the isomorphism $\mathfrak{g}(\mathcal{H}) \cong \mathfrak{so}(1, 4)$ explicitly. Let

$$V = V_0 \oplus V_2 \oplus V_4 \tag{9.4}$$

be a 5-dimensional graded vector space over \mathbb{R} , $\dim V_0 = \dim V_4 = 1$, $\dim V_2 = 3$. Let $(\cdot, \cdot) : V \times V \longrightarrow \mathbb{R}$ be a bilinear symmetric form satisfying the following conditions

- (i) $(\cdot, \cdot)|_{V_2}$ is positive definite.
- (ii) $(V_0, V_2) = (V_4, V_2) = (V_0, V_0) = (V_4, V_4) = 0$.

Such form a exists and is unique up to a graded automorphism. It has signature $(1, 4)$. We define an isomorphism $i : \mathfrak{g}(\mathcal{H}) \longrightarrow \mathfrak{so}(V)$ as follows.

Let $V^o \subset H^2(M, \mathbb{R})$ be the space spanned by $\omega_I, \omega_J, \omega_K$. Let

$$(\cdot, \cdot)_{\mathcal{H}} : H^2(M) \times H^2(M) \longrightarrow \mathbb{R}$$

be the natural (Bogomolov–Beauville) pairing considered in Section 7. By definition (see [V]), the form $(\cdot, \cdot)_{\mathcal{H}}$ is positive definite on V^o . We identify V_2 with V^o . Let \mathbb{I} be a generator of V_0 and Υ be a generator of V_4 , such that $(\mathbb{I}, \Upsilon) = 1$, α be an index which runs through I, J, K , and ω be an arbitrary vector in V^o . We define an action of $L_{\omega_\alpha}, \Lambda_{\omega_\alpha}$ on V as follows.

$$\begin{aligned}
\text{(i)} \quad & L_{\omega_\alpha} V_4 = \Lambda_{\omega_\alpha} V_0 = 0 \\
\text{(ii)} \quad & L_{\omega_\alpha} \omega = (\omega_\alpha, \omega) \Upsilon \\
\text{(iii)} \quad & \Lambda_{\omega_\alpha} \omega = (\omega_\alpha, \omega) \mathbb{I} \\
\text{(iv)} \quad & L_{\omega_\alpha} \mathbb{I} = \omega_\alpha, \quad \Lambda_{\omega_\alpha} \Upsilon = \omega_\alpha.
\end{aligned} \tag{9.5}$$

An easy argument (see, e. g., [V], Theorem 8.1) implies that (9.5) defines an action of $\mathfrak{g}(\mathcal{H})$ on V .³

Let $\Omega \in V^o \otimes \mathbb{C}$ be the form $\Omega = \omega_J + \sqrt{-1}\omega_K$, and $\Theta_0 : V \otimes \mathbb{C} \longrightarrow V \otimes \mathbb{C}$ be the endomorphism defined by

$$\Theta_0(\mathbb{I}) = \Omega, \quad \Theta_0(\bar{\Omega}) = \Upsilon, \quad \Theta_0(\omega_I) = \omega_I, \quad \Theta_0(\Omega) = \mathbb{I}, \quad \Theta_0(\Upsilon) = \bar{\Omega}.$$

Clearly, Θ_0 is orthogonal and $\det \Theta_0 = 1$. Since $(G(\mathcal{H}) \otimes \mathbb{C})/Z = SO(5, \mathbb{C})$, every orthogonal automorphism γ of $V \times \mathbb{C}$, $\det \gamma = 1$ corresponds to an element of $G(\mathcal{H}, \mathbb{C})/Z$. Therefore, we may consider Θ_0 as an element of $G(\mathcal{H})/Z = \text{Aut}(G) = SO(1, 4)$.

Claim 9.3: The adjoint action of Θ_0 on $\mathfrak{g}(\mathcal{H})$ satisfies (9.3).

Proof: A calculation. ■

It is also possible to check by calculation that η acts on $H^*(M)$ normalizing $\mathfrak{g}(\mathcal{H})$ and the resulting automorphism of $\mathfrak{g}(\mathcal{H})$ coincides with Θ_0 (see Proposition 9.7). However, $\text{Aut}(\mathfrak{g}(\mathcal{H})) \cong \text{Sp}(1, 1)/(\pm 1)$, while $G(\mathcal{H}) \cong \text{Sp}(1, 1)$, unless there is no odd-dimensional cohomology. To give an explicit statement of Theorem 9.1, we need to get rid of the ambiguity in sign.

Let T be a quaternionic-Hermitian space, $\dim_{\mathbb{H}} T = 1$, and $W = \Lambda_{\mathbb{R}}^{\text{odd}}(T)$ be the representation of $\text{Sp}(1, 1)$ defined in the proof of Corollary 8.2. Let $L_{\Omega}, \Lambda_{\Omega} : W \otimes \mathbb{C} \longrightarrow W \otimes \mathbb{C}$, $L_{\Omega} = L_{\omega_J} + \sqrt{-1}L_{\omega_K}$, $\Lambda_{\Omega} = \Lambda_{\omega_J} + \sqrt{-1}\Lambda_{\omega_K}$. Let $L_{\bar{\Omega}}, \Lambda_{\bar{\Omega}} : W \otimes \mathbb{C} \longrightarrow W \otimes \mathbb{C}$ be the complex conjugate operators to $L_{\Omega}, \Lambda_{\Omega}$, $L_{\bar{\Omega}} = L_{\omega_J} - \sqrt{-1}L_{\omega_K}$, $\Lambda_{\bar{\Omega}} = \Lambda_{\omega_J} - \sqrt{-1}\Lambda_{\omega_K}$. Let $\mathfrak{M}_{\Omega}, \mathfrak{M}_{\bar{\Omega}} : W \otimes \mathbb{C} \longrightarrow W \otimes \mathbb{C}$ be the maps

$$t \xrightarrow{\mathfrak{M}_{\Omega}} (L_{\Omega} + \Lambda_{\bar{\Omega}})t,$$

$$t \xrightarrow{\mathfrak{M}_{\bar{\Omega}}} (L_{\bar{\Omega}} + \Lambda_{\Omega})t.$$

Let $W = W^a \oplus W^b$ be the decomposition defined by $W^a = \Lambda^{1,0}(T) \oplus \Lambda^{1,2}(T)$, $W^b = \Lambda^{0,1}(T) \oplus \Lambda^{2,1}(T)$. Since $\Omega = \omega_J + \sqrt{-1}\omega_K$ is a $(2,0)$ -form, and $\bar{\Omega}$ is a $(0,2)$ -form,

³Let T be a quaternionic-Hermitian space, $\dim_{\mathbb{H}} T = 1$. As a representation of $\mathfrak{g}(\mathcal{H}) = \mathfrak{g}(T)$, the space V is canonically isomorphic to $\mathfrak{g}(T)\Lambda^0(T) \subset \Lambda^*(T)$, where $\mathfrak{g}(T)\Lambda^0(T)$ is a minimal $\mathfrak{g}(T)$ -invariant subspace of $\Lambda^*(T)$ containing 0-forms.

- L_Ω maps $\Lambda^{p,q}(T)$ to $\Lambda^{p+2,q}(T)$
- $\Lambda_{\bar{\Omega}}$ maps $\Lambda^{p,q}(T)$ to $\Lambda^{p-2,q}(T)$
- $L_{\bar{\Omega}}$ maps $\Lambda^{p,q}(T)$ to $\Lambda^{p,q+2}(T)$
- Λ_Ω maps $\Lambda^{p,q}(T)$ to $\Lambda^{p,q-2}(T)$.

Therefore, $L_\Omega, \Lambda_{\bar{\Omega}}$ vanish on W^a and $L_{\bar{\Omega}}, \Lambda_\Omega$ vanish on W^b . The triples $L_\Omega, \Lambda_{\bar{\Omega}}, H - \sqrt{-1}ad I$ and $L_{\bar{\Omega}}, \Lambda_\Omega, H + \sqrt{-1}ad I$ satisfy relations of $\mathfrak{sl}(2)$ for the standard vectors $e, f, h \in \mathfrak{sl}(2)$. Clearly, W^b is a representation of weight one for the $\mathfrak{sl}(2)$ generated by $L_\Omega, \Lambda_{\bar{\Omega}}, H - \sqrt{-1}ad I$, and W^a for $L_{\bar{\Omega}}, \Lambda_\Omega, H + \sqrt{-1}ad I$. A computation of the action of $f + e$ on $\mathfrak{sl}(2)$ -representations of weight one implies that $(\mathfrak{M}_\Omega)^2|_{W^b} = Id$ and $(\mathfrak{M}_{\bar{\Omega}})^2|_{W^a} = Id$. Let $\underline{\Theta} \in End(W \otimes \mathbb{C})$,

$$\underline{\Theta}(t) = \mathfrak{M}_\Omega + \mathfrak{M}_{\bar{\Omega}}^2.$$

Lemma 9.4: The endomorphism $\underline{\Theta} \in End(W \otimes \mathbb{C})$ belongs to $Sp(1, 1) \otimes \mathbb{C} \subset End(W \otimes \mathbb{C})$.

Proof: Define a scalar product (\cdot, \cdot) on $\Lambda^{\text{odd}}(T)$ as follows. Let $\varepsilon : \Lambda^*(T) \longrightarrow \mathbb{R}$ be the standard projection to $\Lambda^4(T) \cong \mathbb{R}$, and

$$\langle \cdot, \cdot \rangle : \Lambda^{\text{odd}}(T) \times \Lambda^{\text{odd}}(T) \longrightarrow \mathbb{R}$$

be the Poincaré form defined by $\langle x, y \rangle = \varepsilon(x \wedge y)$. We define

$$\begin{aligned} (x, y) &= \langle x, y \rangle, \quad \text{for } \dim x = 1, \dim y = 3, \\ &= -\langle x, y \rangle, \quad \text{for } \dim x = 3, \dim y = 1. \end{aligned}$$

According to the standard construction of the $Sp(1, 1)$ -action on $\Lambda^{\text{odd}}(T)$ (see [V-so], Appendix, or any standard textbook on classical groups, like [VO]), the group $Sp(1, 1)$ is identified with the group of all endomorphisms of W which belong to $End_{\mathbb{H}}(W)$, preserve the scalar product (\cdot, \cdot) and have determinant 1. From construction, it is clear that $\underline{\Theta} \in End_{\mathbb{H}}(W) \otimes \mathbb{C}$. The eigenvalues of $\underline{\Theta}$ are -1 of multiplicity 2 and 1 of multiplicity 6. Therefore, $\det \underline{\Theta} = 1$. The decomposition $W = W^a \oplus W^b$ is orthogonal, and $\underline{\Theta}$ is the identity on W^a . Therefore, to prove that $\underline{\Theta} \in G(T, \mathbb{C})$ it remains to show that $\underline{\Theta}|_{W^b}$ preserves the scalar product in W^b . Let z_1, z_2 be an orthonormal basis in $\Lambda^{1,0}(T)$. Then W^b is spanned by $e_1^1 := \bar{z}_1, e_1^2 := \bar{z}_2, e_1^3 := \bar{z}_1 \wedge \Omega,$

$e_2^2 := \bar{z}_2 \wedge \Omega$, where $\Omega = z_1 \wedge z_2$. Let $\mathfrak{d}(i, j)$ equal 1 if either $i = 1, j = 2$ or $i = 2, j = 1$ and 0 otherwise. Let $p : \{1, 2\} \longrightarrow \{1, 2\}$ be the function defined by $p(2) = 1, p(1) = 2$. As the definition implies,

$$(e_i^j, e_k^l) = \mathfrak{d}(i, k)\mathfrak{d}(j, l).$$

and $\mathfrak{M}_\Omega(e_i^j) = e_{p(i)}^j$. This implies that $\underline{\Theta}|_{W^b}$ is orthogonal. Lemma 9.4 is proven. ■

Let

$$i : G(T, \mathbb{C}) \longrightarrow G(\mathcal{H}, \mathbb{C}), \quad G(T) = Sp(1, 1)$$

be the natural homomorphism.⁴ Denote the element $i(\underline{\Theta}) \in G(\mathcal{H}, \mathbb{C})$ by Θ . The precise version of Theorem 9.1 follows.

Theorem 9.1' Consider η, Θ as endomorphisms of $H^*(M, \mathbb{C})$. Then

$$\eta = \Theta. \tag{9.6}$$

Proof: Here is the plan of the proof. As in the proof of Theorem 8.1, we reduce (9.6) to the similar equation which holds for exterior forms over a quaternionic-Hermitian space. The Serre duality operator can be defined on forms, and then it is shown that it carries over to harmonic forms, which are the cohomology. We can define the “formal” Serre duality operator

$$\eta^{p,q} : \Lambda_I^{p,q}(T) \longrightarrow \Lambda_I^{2n-p,q}(T),$$

where T is a quaternionic-Hermitian space, $\dim_{\mathbb{H}} T = n$. The group $G(\mathcal{H}, \mathbb{C})$ acts on $\Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C}$. Therefore, we can speak of action of Θ on this space. Now, to prove Theorem 9.1' it will suffice to prove (9.6) for η, Θ acting in $\Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C}$.

Let T be a quaternionic-Hermitian space, and $\Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C} = \oplus \Lambda_I^{p,q}(T)$ be the Hodge decomposition corresponding to an operator $I \in \mathbb{H}$ considered as a complex structure on T . We define the Serre duality operator

$$\eta^{p,q} : \Lambda_I^{p,q}(T) \longrightarrow \Lambda_I^{2n-p,q}(T)$$

⁴Which is an isomorphism unless $H^{\text{odd}}(M)$ is empty.

as follows. Let $\Omega \in \Lambda^{2,0}(T)$, $\Omega = \omega_J + \sqrt{-1} \omega_K$. Then Ω defines a non-degenerate \mathbb{C} -linear pairing $\Lambda_I^{1,0}(T) \times \Lambda_I^{1,0}(T) \longrightarrow \mathbb{C}$. By multiplicativity, this defines a non-degenerate pairing

$$\Lambda_I^{i,0}(T) \times \Lambda_I^{i,0}(T) \longrightarrow \mathbb{C}. \quad (9.7)$$

Let $n = \dim_{\mathbb{H}}(T)$. Then the n -th exterior power of Ω is a non-zero vector in the one-dimensional space $\Lambda_I^{2n,0}(T)$. Using the form Ω^n , we define a non-degenerate pairing

$$\Lambda_I^{i,0}(T) \times \Lambda_I^{2n-i,0}(T) \longrightarrow \mathbb{C} \quad (9.8)$$

Considering (9.7), (9.8) as isomorphisms between $\Lambda^{*,0}(T)$ and $(\Lambda^{*,0}(T))^*$, we can form the composition of (9.7) and (9.8), which is a map

$$\eta_T^{i,0} : \Lambda^{i,0}(T) \longrightarrow \Lambda^{2n-i,0}(T).$$

Using the canonical isomorphism $\Lambda^{p,q} \cong \Lambda^{p,0} \otimes \Lambda^{0,q}$, we define

$$\eta_T^{p,q} : \Lambda^{p,q}(T) \longrightarrow \Lambda^{2n-p,q}(T), \quad \eta_T^{p,q}(\omega^{p,0} \otimes \omega^{0,q}) := \eta_T^{p,0}(\omega^{p,0}) \otimes \omega^{0,q}.$$

Let

$$\eta_T := \oplus \eta_T^{p,q} : \Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C} \longrightarrow \Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C}. \quad (9.9)$$

Proposition 9.5: Let M be a compact hyperkähler manifold, of complex dimension $2n$. Let $\eta : H_I^{p,q}(M) \longrightarrow H_I^{2n-p,q}(M)$ be the Serre duality operator associated with the complex structure I and a holomorphic symplectic form $\Omega = \omega_J + \sqrt{-1} \omega_K$. Consider the tangent bundle of M as a bundle of quaternionic-Hermitian spaces, and let $\eta_T : \Lambda_{\mathbb{R}}^*(M) \otimes \mathbb{C} \longrightarrow \Lambda_{\mathbb{R}}^*(M) \otimes \mathbb{C}$ be the operator defined fiberwise as in (9.9). Then η_T preserves harmonic forms, and the resulting endomorphism of $H^*(M, \mathbb{C})$ coincides with η .

Proof: Follows from the definition. ■

Return to the proof of Theorem 9.1'. Let T be a quaternionic-Hermitian space and η_T, Θ be the endomorphisms of $\Lambda_{\mathbb{R}}^*(T) \otimes \mathbb{C}$ constructed above. By Proposition 9.5, we need to show that $\eta_T = \Theta$. Let $T = \oplus T_i$ be a decomposition of T to a direct sum of quaternionic-Hermitian subspaces.

The operators η_T and Θ are multiplicative with respect to this decomposition (for η_T it follows from the definition, and for Θ from (8.1)):

$$\begin{aligned} \forall t \in \Lambda^*(T), t = \otimes t_i, t_i \in \Lambda^*(T_i), \\ \eta_T(t) = \eta_T(t_1) \otimes \eta_T(t_2) \otimes \dots \otimes \eta_T(t_n), \\ \Theta(t) = \Theta(t_1) \otimes \Theta(t_2) \otimes \dots \otimes \Theta(t_n) \end{aligned} \quad (9.10)$$

Therefore, we may assume that $\dim_{\mathbb{H}} T = 1$. On $\Lambda^{\text{odd}}(T)$, we have written the action of Θ explicitly in coordinates (Lemma 9.4). It is clear that η acts on $\Lambda^{\text{odd}}(T)$ exactly in the same way.

Let $G_o = SU(2)$ be the group of unitary quaternions acting on $\Lambda^*(T)$ by multiplication, and $\Lambda^-(T)$ be the space of all G_o -invariant 2-forms. Consider $\Lambda^{\text{even}}(T)$ as a representation of $\mathfrak{g}(T)$. It is easy to check that $\Lambda^{\text{even}}(T)$ decomposes as

$$\Lambda^{\text{even}}(T) = \Lambda^-(T) \oplus \left(\mathfrak{g}(T) \cdot \Lambda^0(T) \right),$$

where

$$\dim \Lambda^-(T) = 3, \quad \dim \left(\mathfrak{g}(T) \cdot \Lambda^0(T) \right) = 5.$$

On the space $\Lambda^-(T)$, the Lie algebra $\mathfrak{g}(T)$ acts trivially, and therefore, Θ acts as the identity. Since $\dim_{\mathbb{H}} T = 1$, η acts as the identity on $(1, 0)$ -forms. Therefore, η acts as the identity on $\Lambda^-(T)$. As a representation of $\mathfrak{g}(T)$, the space $\mathfrak{g}(T) \cdot \Lambda^0(T) \subset \Lambda^{\text{even}}(T)$ is isomorphic to V of (9.4). We have written explicitly the action of Θ on V . It is easy to check that η preserves $\mathfrak{g}(T) \cdot \Lambda^0(T)$ and its action on $\mathfrak{g}(T) \cdot \Lambda^0(T)$ coincides with Θ . This proves Theorem 9.1. ■

Appendix. Here we give a summary of the interaction between η and the standard Hodge operators acting on the cohomology of M .

Let M be a compact Kähler manifold. For a Kähler class $\omega \in H_I^{1,1}(M)$, consider the Hodge endomorphism $\Lambda_\omega : H^i(M) \longrightarrow H^{i-2}(M)$. A well-known linear algebra argument⁵ implies that the operator Λ_ω depends only on the cohomology class ω , and is independent of the Riemannian or complex structure on M . Let $S \subset H^2(M, \mathbb{R})$ be the set of all ω for which there exists a

⁵[BLie], VIII § 11; see also [V], Proposition 7.1.

complex structure I for which ω lies in Kähler cone. If M is holomorphically symplectic, [V], Lemma 12.1 implies that for all a, b , for which $a, b, a + b$ lie in S , we have

$$(a + b, a + b)_{\mathcal{H}} \Lambda_{a+b} = (a, a)_{\mathcal{H}} \Lambda_a + (b, b)_{\mathcal{H}} \Lambda_b.$$

In [V], Lemma 5.6, we proved that S is open in $H^2(M, \mathbb{R})$. From the argument used in the proof of Lemma 5.6, [V], it is clear that for every open set $U \subset H^2(M, \mathbb{R})$, there exists $x \in S$ such that for all $y \in U$, we have $x + y \in S$. Therefore, there exist an (obviously, unique) linear map

$$\check{\Lambda} : H^2(M) \longrightarrow \text{End}(H^*(M))$$

such that for all $a \in S$, we have

$$\check{\Lambda}(a) = (a, a)_{\mathcal{H}} \Lambda_a.$$

Definition 9.6: Throughout this paper, we set Λ_a to be $(a, a)_{\mathcal{H}}^{-1} \check{\Lambda}(a) \in \text{End}(H^*(M))$ for all $a \in H^2(M, \mathbb{C})$, $(a, a)_{\mathcal{H}} \neq 0$.

Proposition 9.7: Let M be a holomorphically symplectic manifold of Kähler type, Ω a holomorphic symplectic form, η the operator of Serre duality, ω a class in $H^{1,1}(M)$. For each automorphism $x \in \text{End}(H^*(M))$, denote by $\eta(x)$ the endomorphism $\eta x \eta^{-1}$. Then

- (i) η is an involution.
- (ii) $\eta(L_\omega) = [\Lambda_\Omega, L_\omega]$.
- (iii) $\eta(\Lambda_\omega) = [\Lambda_{\bar{\Omega}}, L_\omega]$.
- (iv) $\eta(L_{\bar{\Omega}}) = \Lambda_\Omega$, $\eta(L_{\bar{\Omega}}) = L_\Omega$, $\eta(\Lambda_{\bar{\Omega}}) = \Lambda_\Omega$,

where $L_a : H^i(M) \longrightarrow H^{i+2}(M)$ is the operator of multiplication by $a \in H^2(M)$, and $\Lambda_\Omega, \Lambda_{\bar{\Omega}}$ are the operators defined above.

Proof: The part (i) follows from the definition (compare η with the Hodge operator $*$, which is also involutive). To prove (ii) – (iii), notice that the Kähler cone is Zariski dense in $H^{1,1}(M)$. Therefore, we may assume that ω is a Kähler class. Then, (ii), (iii) is a consequence of standard identities in the algebra $\mathfrak{g}(\mathcal{H}) \cong \mathfrak{so}(1, 4)$ associated with the hyperkähler structure. Part (iv) is straightforward. ■

10 Yukawa VFA as an equivariant VFA (proofs).

The chief tool used in our calculations is the following theorem.

Theorem 10.1: ([V], Theorem 11.1) Let M be a compact holomorphically symplectic manifold. Assume that $\dim H^{2,0}(M) = 1$. Let \mathbb{V} be the linear space $H^2(M, \mathbb{R})$ equipped with a natural scalar product $(\cdot, \cdot)_{\mathcal{H}}$ of [Beau], Remarques, p. 775, and [V], Theorem 5.1. Let $\mathfrak{g}(M) \subset \text{End}(H^*(M))$ be the Lie algebra generated by $L_{\omega}, \Lambda_{\omega}$, where ω runs through Kähler classes corresponding to all complex structures on M . Then $\mathfrak{g}(M)$ is isomorphic to $\mathfrak{so}(\mathbb{V} \oplus \mathfrak{H})$, where \mathfrak{H} is a 2-dimensional real vector space with hyperbolic quadratic form.¹

■

Let $G(M) \subset \text{End}(H^*(M))$ be the Lie group corresponding to $\mathfrak{g}(M)$.

Lemma 10.2: Let M be a hyperkähler manifold equipped with a hyperkähler structure $(I, J, K, (\cdot, \cdot))$, and $\Theta \in G(\mathcal{H}) \subset G(M)$ be the group element constructed in Section 9. Let

$$\omega_1, \omega_2 \in H_I^{1,1}(M), \quad (\omega_1, \omega_I)_{\mathcal{H}} = (\omega_2, \omega_I)_{\mathcal{H}} = (\omega_1, \omega_2)_{\mathcal{H}} = 0. \quad (10.1)$$

Consider $g = [L_{\omega_1}, \Lambda_{\omega_2}]$ as an element of the Lie algebra $\mathfrak{g}_0(M) \subset \mathfrak{g}(M)$. Then Θ stabilizes g :

$$\text{Ad}(\Theta)g = g.$$

Proof: Let $E_g \subset G(M) \otimes \mathbb{C}$ be the one-dimensional subgroup corresponding to g . We need to show that for all $\gamma \in E_g$, γ commutes with Θ . By [V], Proposition 13.2, for all $\omega \in H^2(M)$, we have

$$[g, L_{\omega}] = (\omega, \omega_1)_{\mathcal{H}} L_{\omega_2} - (\omega, \omega_1)_{\mathcal{H}} L_{\omega_1},$$

and

$$[g, \Lambda_{\omega}] = -(\omega, \omega_1)_{\mathcal{H}} \Lambda_{\omega_2} + (\omega, \omega_1)_{\mathcal{H}} \Lambda_{\omega_1}.$$

¹For a holomorphically symplectic manifold with $h^{2,0}(M) = 1$, $h^1(M) = 0$, the signature of $(\cdot, \cdot)_{\mathcal{H}}$ is $(3, m-3)$, for $m = h^2(M)$ (see [V]). In this case, the Lie algebra $\mathfrak{g}(M)$ is isomorphic to $\mathfrak{so}(4, m-2)$.

Since $H_I^{2,0}(M)$ is orthogonal to $H_I^{1,1}(M)$, we obtain that

$$[g, L_{\omega_\alpha}] = [g, \Lambda_{\omega_\alpha}] = 0,$$

for $\alpha = J, K$. Since ω_i are orthogonal to ω_I , we know that $[g, L_{\omega_I}] = [g, \Lambda_{\omega_I}] = 0$. Therefore, for $\gamma \in E_g$, the automorphism

$$Ad(\gamma) : \mathfrak{g}(M) \otimes \mathbb{C} \longrightarrow \mathfrak{g}(M) \otimes \mathbb{C}$$

acts trivially on $\mathfrak{g}(\mathcal{H}) \otimes \mathbb{C} \subset \mathfrak{g}(M) \otimes \mathbb{C}$. Therefore, γ commutes with Θ . This proves Lemma 10.2. ■

Proposition 10.3: Let M be a holomorphically symplectic manifold, and (I_1, Ω_1) , (I_2, Ω_2) holomorphic symplectic structures. Let

$$\eta_1, \eta_2 : H^*(M) \longrightarrow H^*(M)$$

be the Serre duality operators corresponding to

$$(I_1, \Omega_1), (I_2, \Omega_2).$$

Assume that the cohomology classes $[\Omega_1], [\Omega_2] \in H^2(M, \mathbb{C})$ are equal. Then $\eta_1 = \eta_2$.

Proof: Let $\mathcal{H}_1, \mathcal{H}_2$ be the hyperkähler structures which induce (I_1, Ω_1) , (I_2, Ω_2) respectively. Let $\omega_\alpha^i \in H^2(M, \mathbb{R})$, $\alpha = I, J, K$, $i = 1, 2$ be the standard cohomology classes $\omega_I, \omega_J, \omega_K$ for the hyperkähler structure \mathcal{H}_i . Since $[\Omega_1] = [\Omega_2]$, we have $\omega_K^1 = \omega_K^2$ and $\omega_J^1 = \omega_J^2$. If $\omega_I^1 = \omega_I^2$, then the Lie algebras $\mathfrak{g}(\mathcal{H}_1), \mathfrak{g}(\mathcal{H}_2) \subset \text{End}(H^*(M))$ coincide. Therefore, the groups $G(\mathcal{H}_1), G(\mathcal{H}_2)$ also coincide. Expressing η_i in terms of $G(\mathcal{H}_i)$ as in (9.6), we obtain that $\eta_1 = \eta_2$.

Using Theorem 7.1, it is easy to show that the Hodge decompositions $H_I^{p,q}(M) := H_{I_1}^{p,q}(M), H_{I_2}^{p,q}(M)$ coincide. Let K_i , $i = 1, 2$ be the Kähler cones of the complex structures I_i . If K_1 intersects K_2 , we may apply Calabi-Yau to find the hyperkähler structures $\mathcal{H}_1, \mathcal{H}_2$ satisfying $\omega_I^1 = \omega_I^2$, such that \mathcal{H}_i induces (I_i, Ω_i) , $i = 1, 2$. Applying the previous argument, we find that in this case $\eta_1 = \eta_2$. However, it is *a priori* possible that $K_1 \cap K_2 = \emptyset$. In this case, we apply the following argument.

Let $\omega_1, \omega_2, \omega_3$ be classes in $H^2(M, \mathbb{R})$ satisfying

$$(\omega_1, \omega_1)_{\mathcal{H}} = (\omega_2, \omega_2)_{\mathcal{H}} = (\omega_3, \omega_3)_{\mathcal{H}} > 0, (\omega_i, \omega_j)_{\mathcal{H}} = 0. \quad (10.2)$$

Let $P \subset H^2(M, \mathbb{R})$ be the linear span of ω_i , $i = 1, 2, 3$. Let $\mathfrak{g}(P) \subset \mathfrak{g}(M)$ be the Lie algebra generated by L_ω , Λ_ω , for $\omega \in P$. The argument proving Theorem 10.1 easily implies that $\mathfrak{g}(P)$ is isomorphic to $\mathfrak{so}(1, 4)$ in the same way as $\mathfrak{g}(\mathcal{H})$ is. Applying the standard construction, we obtain a Serre duality operator $\Theta_{\omega_1, \omega_2, \omega_3} \in \text{End}(H^*(M))$ which is defined in the same way as Θ for $\mathfrak{g}(\mathcal{H})$.

Lemma 10.4: Under assumptions of Proposition 10.3, let S be the set of all cohomology classes $\omega \in H_I^{1,1}(M)$ satisfying

$$(\omega, \omega)_{\mathcal{H}} = (\omega_J, \omega_J)_{\mathcal{H}}, \quad (\omega, \omega_J)_{\mathcal{H}} = (\omega, \omega_K)_{\mathcal{H}} = 0, i \neq j. \quad (10.3)$$

Let $\omega, \omega' \in S$. Then $\Theta_{\omega, \omega_J, \omega_K} = \Theta_{\omega', \omega_J, \omega_K}$.

Proof: For $\omega, \omega' \in K_1$, this is implied by $\Theta_{\omega, \omega_J, \omega_K} = \eta_1$, $\Theta_{\omega', \omega_J, \omega_K} = \eta_1$, which is a consequence of Theorem 9.1'. (see the argument in the beginning of Proposition 10.3, Proof) Clearly, the set S is connected, and $S \cap K_1$ is Zariski dense in S . Since $\Theta_{\omega, \omega_J, \omega_K}$ is expressed algebraically as a function of ω , and is constant in a Zariski dense set, the map

$$\Theta_{\cdot, \omega_J, \omega_K} : S \longrightarrow \text{End}(H^*(M))$$

is constant. This proves Lemma 10.4 and Proposition 10.3 ■

To prove Theorem 7.3, Theorem 7.4, we use the following claim:

Claim 10.5: Let $\mathcal{H}_1, \mathcal{H}_2$ be hyperkähler structures on M , and η_1, η_2 be the corresponding Serre duality operators. Denote by ω_α^i , $\alpha = I, J, K$, $i = 1, 2$ the natural Kähler forms associated with $\mathcal{H}_1, \mathcal{H}_2$. Let $g \in G_0(M)$ be an element such that under the natural action of $G_0(M)$ on $H^2(M)$, $g(\omega_\alpha^1) = \omega_\alpha^2$. Then $g\eta_1g^{-1} = \eta_2$.

Proof: Follows from (9.6). ■

Now, Theorem 7.3 is equivalent to the following statement:

Theorem 7.3': Let M be a holomorphically symplectic manifold, $\dim_{\mathbb{C}} M = n$, $[\Omega] \in H^2(M, \mathbb{C})$ be the cohomology class of its holomorphic symplectic form, $g \in G_0(M)$ be an element preserving the line passing through Ω .² Using Lemma 10.4 which maps $[\Omega]$ to $c[\Omega]$, $c \in \mathbb{C}$, we

²By definition of $G_0^I(M)$, the group element g preserves the line passing through $[\Omega]$ if and only if $g \in G_0^I(M)$.

may canonically associate the Yukawa product map on $H^*(M)$ with a cohomology class $\Omega_0 \in H^2(M, \mathbb{C})$, given that Ω_0 is the class of a holomorphic symplectic form for some complex structure on M . Let $Y_1, Y_2 : H^*(M) \times H^*(M) \longrightarrow H^*(M)$ be the Yukawa products associated with the cohomology classes $[\Omega]$, $c[\Omega] \in H_I^{2n,0}(M)$. Then

$$gY_1g^{-1} = Y_2.$$

Proof: Let $\eta_\Omega, \eta_{\Omega'}$ be the operators η associated with Ω, Ω' , where $\Omega' = c\Omega$. Claim 10.5 implies that $\eta_{\Omega'} = g\eta_\Omega g^{-1}$. Now,

$$Y_1(x, y) = \eta_\Omega^{-1}(\eta_\Omega(x) \cup \eta_\Omega(y))$$

and

$$Y_2(x, y) = \eta_{\Omega'}^{-1}(\eta_{\Omega'}(x) \cup \eta_{\Omega'}(y)).$$

Since $g \in G_0(M)$, the group element g acts by automorphisms on the (usual) cohomology ring. Therefore,

$$\begin{aligned} Y_2(x, y) &= g\eta_\Omega^{-1}g^{-1}(g\eta_\Omega g^{-1}(x) \cup g\eta_\Omega g^{-1}(y)) \\ &= g\eta_\Omega^{-1}(\eta_\Omega g^{-1}(x) \cup \eta_\Omega g^{-1}(y)) \\ &= gY_1(g^{-1}x, g^{-1}y). \end{aligned}$$

Theorem 7.3' is proven. ■

Proof of Theorem 7.4: Let

$$\bullet_{I_1}, \bullet_{I_2} : H^*(M) \times H^*(M) \longrightarrow H^*(M) \otimes H^{n,0}(M)$$

be the Yukawa multiplication maps constructed by $I_1, I_2 \in \text{Comp}$,

$$H^{n,0}(M) = \underline{K}|_{I_1} = \underline{K}|_{I_2}.$$

Proposition 10.3 implies that, whenever $P_o(I_1) = P_o(I_2)$, we have $\bullet_{I_1} = \bullet_{I_2}$. Therefore, there exist a tensor $\underline{\bullet}_Y : B \times B \longrightarrow B \otimes K$ such that \bullet_Y is obtained as a pullback of Y .

To prove Theorem 7.4, it remains to show that $\underline{\bullet}_Y$ is equivariant with respect to the natural $G_0(M)$ -equivariant structure on B . This is implied by Claim 10.5. ■

11 Periods of hyperkähler manifolds and Tian – Todorov coordinates.

Let M be a holomorphically symplectic manifold, and $Comp$ be its (marked, coarse) deformation space. We give the following description of Tian–Todorov coordinates on $Comp$.

Let $\mathfrak{g}_0(M) \subset \text{End}(H^*(M))$ be the Lie algebra of Theorem 7.1,

$$\mathfrak{g}_0(M) \cong \mathfrak{so}(H^2(M, \mathbb{R}), (\cdot, \cdot)_{\mathcal{H}}).$$

Let $I \in Comp$, and $ad I \in \mathfrak{g}_0(M)$ be the corersponding endomorphism of cohomology. Let $G_0(M) \subset \text{End}(H^*(M))$ be the Lie group corresponding to $\mathfrak{g}_0(M)$ and $X \subset \mathfrak{g}_0(M)$, $X = G_0(M) \cdot ad I$ be the adjoint orbit containing $ad I$. As we have seen previously, the map

$$P_o : Comp \longrightarrow X, \quad J \longrightarrow ad J$$

is étale.

Let

$$\mathfrak{g}_0(M) \otimes \mathbb{C} = \mathfrak{g}_0^{I,-2}(M) \oplus \mathfrak{g}_0^{I,0}(M) \oplus \mathfrak{g}_0^{I,2}(M) \quad (11.1)$$

be the decomposition of $\mathfrak{g}_0(M) \otimes \mathbb{C}$ defined by $[ad I, g] = \sqrt{-1} kg$, for all $g \in \mathfrak{g}_0^{I,k}(M)$, $k = -2, 0, 2$. Earlier, we used the notation $\mathfrak{g}_0^I(M)$ for $\mathfrak{g}_0^{I,0}(M) \cap \mathfrak{g}_0(M)$.

Claim 11.1: There is a natural isomorphism of linear spaces

$$\mathfrak{g}_0^{I,-2}(M) \cong \mathfrak{g}_0^{I,2}(M) \cong H^{1,1}(M),$$

provided by the maps

$$\alpha \longrightarrow [L_\alpha \Lambda_\Omega] \in \mathfrak{g}_0^{I,-2}(M), \quad \alpha \longrightarrow [L_\alpha \Lambda_{\bar{\Omega}}] \in \mathfrak{g}_0^{I,2}(M),$$

for all $\alpha \in H^{1,1}(M)$.

Proof: Let $\mathfrak{g}(M)$ be the Lie algebra of Theorem 10.1. The space $H^*(M)$ is graded by the dimensions of cocycles. Consider the associated grading $\mathfrak{g}(M) = \mathfrak{g}_{-2}(M) \oplus \mathfrak{g}_0(M) \oplus \mathfrak{g}_2(M)$. In [V], we proved that $\mathfrak{g}_2(M)$ is the linear span of all operators L_ω , for all $\omega \in H^2(M, \mathbb{R})$, and $\mathfrak{g}_{-2}(M)$ is the linear span of all Λ_ω , $\omega \in H^2(M, \mathbb{R})$ for all ω such that Λ_ω is defined. Let $\mathfrak{g}_i^{I,j}(M) \subset \mathfrak{g}_i(M) \otimes \mathbb{C}$ be the space of all $x \in \mathfrak{g}_i(M)$ satisfying $[ad I, x] = j\sqrt{-1}x$. Clearly,

$\mathfrak{g}_2^{I,0}(M) \otimes \mathbb{C}$ is the linear span of all operators $L_\omega : H^*(M) \longrightarrow H^*(M)$, $\omega \in H^{1,1}(M)$. Similarly, $\mathfrak{g}_{-2}^{I,0}(M) \otimes \mathbb{C}$ is the linear span of all Λ_ω , $\omega \in H^{1,1}(M)$, such that Λ_ω is defined. This identifies $\mathfrak{g}_{\pm 2}^{I,0}(M)$ with $H^{1,1}(M)$. As we show in Section 9, the adjoint action of η interchanges $\sqrt{-1}H$ and $ad I$, where H is the standard Hodge operator, $H = [L_\omega, \Lambda_\omega]$. Therefore, η identifies $\mathfrak{g}_i^{I,j}(M)$ with $\mathfrak{g}_j^{I,i}(M)$. Applying this to $\mathfrak{g}_{\pm 2}^{I,0}(M) \cong H^{1,1}(M)$, we obtain an identification of $\mathfrak{g}_0^{I,\pm 2}(M)$ with $H^{1,1}(M)$. Unravelling this definition and applying Proposition 9.7, we obtain Claim 11.1. ■

Let $R : X \longrightarrow \mathbb{P}(H^2(M, \mathbb{C}))$ be the map associating a line

$$H_I^{2,0}(M) = \{ \lambda \in H^2(M, \mathbb{C}) \mid ad I(x) = 2\sqrt{-1}x \}$$

to $ad I \in X \subset G_0(M)$.

Lemma 11.2: The map $R : X \longrightarrow \mathbb{P}(H^2(M, \mathbb{C}))$ is a closed, complex analytic embedding. The image of R coincides with the conic $C \subset \mathbb{P}(H^2(M, \mathbb{C}))$,

$$C = \{ l \in \mathbb{P}(H^2(M, \mathbb{C})) \mid (l, l)_\mathcal{H} = 0 \}$$

where $(\cdot, \cdot)_\mathcal{H}$ is the canonical scalar product on $H^2(M)$ (Section 7).

Proof: Follows from a trivial linear-algebraic argument ([Tod2]; see also [V], Section 6). ■

Let $G_0(M) \subset End(H^*(M))$ be the Lie group associated with $\mathfrak{g}_0(M)$. Consider the action of a group $G_0(M) \subset End(H^*(M))$ corresponding to $\mathfrak{g}_0(M) = \mathfrak{so}\left(H^2(M); (\cdot, \cdot)_\mathcal{H}\right)$ on $H^2(M)$. This defines an action of $G_0(M) \otimes \mathbb{C}$ on $C \subset \mathbb{P}(H^2(M, \mathbb{C}))$.

Consider the action of $\mathfrak{g}^{-2,I}(M) \subset \mathfrak{g}_0(M) \otimes \mathbb{C}$ on $C = X$. The Lie algebra $\mathfrak{g}^{-2,I}(M)$ is commutative. Thus, we obtain a number of commuting holomorphic vector fields on X . Since the action of $G_0(M)$ on X is transitive, $\dim X = \dim \mathfrak{g}^{-2,I}(M) = \dim H^{1,1}(M)$ and $G_0^{2,I}(M)$, $G_0^{0,I}(M)$ stabilizes $R(ad I)$, the action of the abelian Lie group $G^{-2,I}(M)$ defines coordinates in a neighbourhood U of $R(ad I) \in X$. We have obtained a natural open embedding $i : B \hookrightarrow C = X$, where B is an open ball in $\mathfrak{g}_0^{-2,I}(M)$.

Theorem 11.3: Let M be a holomorphically symplectic manifold, X its moduli space and $i : B \hookrightarrow X$ be the open embedding constructed above. Lift i to a map $\tilde{i} : B \hookrightarrow \text{Comp}$. Identifying $\mathfrak{g}_0^{2,I}(M)$ with $H^{1,1}(M)$ as in Claim 11.1, we can realize B as an open ball in $H^{1,1}(M)$. Then, for B sufficiently small, \tilde{i} coincides with the Tian–Todorov map of Theorem 4.5.6.

Proof of Theorem 11.3: Let $\Omega \in H^{2,0}(M, \mathbb{C})$ be the holomorphic symplectic form of M , $v \in B \subset \mathfrak{g}_0^{-2,I}(M)$ and $e^v(\Omega) \in C$ be the point of C corresponding to v . Let $P_c : \text{Comp} \rightarrow C$ be the map associating the line $H_L^{2,0}(M) \in \mathbb{P}(H^2(M, \mathbb{C}))$ to the complex structure $L \in \text{Comp}$. Let $\bar{\Omega} \in \Lambda^{0,2}(M)$ be the complex conjugate to Ω . The standard identification of $H^{1,1}(M)$ with $\mathfrak{g}_0^{-2,I}$ goes as

$$\lambda \xrightarrow{\tau} [L_\lambda, \Lambda_{\bar{\Omega}}],$$

where $\lambda \in H^{1,1}(M)$ and $L_\lambda, \Lambda_{\bar{\Omega}} \in \text{End}(H^*(M))$ are the standard Hodge operators of $[V]^1$ associated with $\lambda, \bar{\Omega}$. Then Theorem 11.3 is equivalent to the following explicit statement:

$$(e^v(\Omega))^{\mathbb{P}} = P_c(\varphi(\tau^{-1}(v))), \quad (11.2)$$

where $\varphi : B \rightarrow \text{Comp}$ is the Tian–Todorov map, and

$$(\cdot)^{\mathbb{P}} : H^2(M, \mathbb{C}) \setminus 0 \rightarrow \mathbb{P}(H^2(M, \mathbb{C}))$$

is the map associating to a vector $x \in H^2(M, \mathbb{C}) \setminus 0$ the line passing through x .

The Kähler cone K is open in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$, and $H^{1,1}(M) \cap H^2(M, \mathbb{R})$ is the set of fixed points of an anti-complex involution of the affine space $H^{1,1}(M)$. Therefore, every complex analytic map on the open ball B is determined by its values on $K \cap B$. Therefore, in proving (11.2), we may assume that the cohomology class $\omega = \tau^{-1}(v)$ is a Kähler class for some Kähler structure on M .

Let $\mathcal{H} = (I, J, K, (\cdot, \cdot))$ be the hyperkähler structure on M , inducing I, Ω and ω :

$$\omega = \omega_I, \quad \Omega = \omega_J + \sqrt{-1} \omega_K.$$

¹The operator L_λ multiplies $\eta \in H^*(M)$ by λ . The operator Λ_Ω is equal to $\Lambda_{\omega_J} + \sqrt{-1} \Lambda_{\omega_K}$, where $\Lambda_{\omega_J}, \Lambda_{\omega_K}$ are the Hodge operators of complex structures J, K , in a hyperkähler structure $(I, J, K, (\cdot, \cdot))$ on M . See also Definition 9.6.

Lemma 11.4: Let M be a hyperkähler manifold and $\omega = \omega_I \in \Lambda^{1,1}(M)$ be its Kähler form. Let $a = \tilde{\omega}$ be the harmonic section of $\Lambda^{0,1}(TM)$ corresponding to ω . Let $\varphi(t\omega) = \sum a_i$, $a_n = G_{\bar{\partial}}\partial \sum_{i+j=n-1} a_i \bullet_Y a_j$ be the image of ta under the Tian-Todorov map, $t \in \mathbb{R}$. Then $a_n = 0$ for all $n > 0$.

Proof: Clearly, it suffices to show that $a_1 = 0$, where $a_1 = G_{\bar{\partial}}\partial(\tilde{\omega} \bullet_Y \tilde{\omega})$. Consider the standard Serre duality operator $\eta : \Omega^{p,q}(M) \longrightarrow \Omega^{n-p,q}(M)$. By definition, $\tilde{\omega} = \eta(\omega)$, and $x \bullet_Y y = \eta(\eta^{-1}(x) \cup \eta^{-1}(y))$. Therefore,

$$a_1 = G_{\bar{\partial}}\partial\tilde{\omega} \bullet_Y \tilde{\omega} = G_{\bar{\partial}}\partial\eta(\omega \cup \omega).$$

Since ω is a Kähler form, $\omega \cup \omega$ is harmonic (Kodaira). The map η is compatible with the Hermitian structure, and commutes with $\bar{\partial}$. Therefore, η maps harmonic forms to harmonic forms. We obtain that $\eta(\omega \cup \omega)$ is also harmonic, and $\partial\eta(\omega \cup \omega) = 0$. ■

Let $A_{t\omega}$ be $t\omega^\sim$ considered as a differential operator (4.2.3), where

$$t\omega^\sim \in \Lambda^{0,1}(M, TM)$$

is the TM -valued $(0,1)$ -form associated with the $(1,1)$ -differential form

$$t\omega \in \Lambda^{0,1}(\Omega^1(M))$$

via the identification $TM \cong \Omega^1 M$ provided by the holomorphic symplectic structure. We obtain that, for a as in Lemma 11.4 and ∂_{new} as in (4.2.3) the operator ∂_{new} is equal to $\bar{\partial} + A_{t\omega}$. The following proposition computes this differential operator in terms of ∂ and the quaternion action.

Proposition 11.5: Let M be a manifold equipped with a hyperkähler structure $(I, J, K, (\cdot, \cdot))$. Consider M as a Kähler manifold with Kähler structure associated with I and (\cdot, \cdot) . Let $\omega = \omega_I$ be its Kähler form. Consider ω as a section of $\Lambda^{0,1}(\Omega^1(M))$. Let \underline{A} be the section of $\Lambda^{0,1}(M, TM)$ obtained from ω through the isomorphism $TM \cong \Omega^1(M)$ provided by the holomorphically symplectic structure, and $A : C^\infty(M) \longrightarrow \Lambda^{0,1}(M)$ be a differential operator corresponding to \underline{A} as in (4.2.3). Since J anticommutes with I , the operator $J : \Lambda_{\mathbb{R}}^1(M) \longrightarrow \Lambda_{\mathbb{R}}^1(M)$ maps $(1,0)$ -forms to $(0,1)$ -forms. Therefore, we may consider a composition $J \circ \partial$ as a differential operator from functions to $(0,1)$ -forms. Then

$$A = J \circ \partial. \tag{11.3}$$

Proof: Since A and $J \circ \partial$ are operators of the first order, and a hyperkähler manifold is flat up to $O(r^2)$, it suffices to check (11.3) when M is a flat manifold. Assume M is flat. Let x_i, y_i be holomorphic coordinates on M , such that the forms $dx_i, dy_i, \overline{dx_i}, \overline{dy_i}$ constitute an orthonormal basis in the bundle of 1-forms, and the holomorphic symplectic form is written as $\Omega = \sum_i dx_i \wedge dy_i$. Then $\omega = \sum dx_i \wedge \overline{dx_i} + dy_i \wedge \overline{dy_i}$, the operator A is written in coordinates as

$$A(f) = \sum_i \frac{\partial f}{\partial x_i} \overline{dy_i} - \sum_i \frac{\partial f}{\partial y_i} \overline{dx_i}$$

and ∂ as

$$\partial f = \sum_i \frac{\partial f}{\partial x_i} dx_i + \sum_i \frac{\partial f}{\partial y_i} dy_i$$

On the other hand, $J(dx_i) = \overline{dy_i}$, and $J(dy_i) = -\overline{dx_i}$. This proves Proposition 11.5. ■

We obtain the following explicit description of the Tian–Todorov map. Let $\omega \in B \subset H^{1,1}(M)$ be a form such that $\omega = t\omega_I$ for a hyperkähler structure $(I, J, K, (\cdot, \cdot))$. Then the complex structure $\varphi(\omega)$ is defined by the differential operator

$$\bar{\partial}_{new} = \bar{\partial} + tI \circ \partial. \quad (11.4)$$

After appropriate substitutions, (we substitute $\varphi(\alpha)$ for $\partial + tI \circ \partial$ and v for $t\omega_I$) the equation (11.2) takes the form

$$\left(e^{t[L_{\omega_I}, \Lambda_{\Omega}]}(\Omega) \right)^{\mathbb{P}} = P_c(\bar{\partial} + tI \circ \partial), \quad (11.5)$$

where by $P_c(\bar{\partial} + tI \circ \partial)$ we understand the line $H_L^{2,0}(M) \subset H^2(M, \mathbb{C})$ associated with a complex structure L defined by the operator $\bar{\partial}_{new} = \bar{\partial} + tI \circ \partial$. To prove Theorem 11.3, it remains to show that (11.5) is true for all hyperkähler structures.

A calculation using identities (8.2) shows that $[L_{\omega_I}, \Lambda_{\Omega}]\Omega = 2\omega$, and $[L_{\omega_I}, \Lambda_{\Omega}]\omega = \bar{\Omega}$. Therefore,

$$e^{t[L_{\omega_I}, \Lambda_{\Omega}]}(\Omega) = \Omega + 2t\omega + t^2\bar{\Omega} \quad (11.6)$$

The algebra $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \text{Mat}_{\mathbb{C}}(2)$ naturally acts on $\Lambda^1(M)$. We extend this action to an action of the group $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^* \cong GL(2, \mathbb{C})$ on $\Lambda^*(M)$.

Claim 11.6: For each $\alpha \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$, the operator

$$\alpha(\bar{\partial} + A_{t\omega}) : f \in C^\infty(M) \longrightarrow \alpha(\bar{\partial} + A_{t\omega}) \in \Lambda^1(M)$$

induces the same holomorphic structure on M as $\bar{\partial} + A_{t\omega}$.

Proof: Let \mathfrak{C} be the kernel of $\bar{\partial} + A : C^\infty(M) \longrightarrow \Lambda^1(M)$. By definition, the function $f \in C^\infty(M)$ is holomorphic with respect to the complex structure defined by $\bar{\partial} + A$ if and only if f lies in \mathfrak{C} . On the other hand, $\ker(\alpha(\bar{\partial} + A_{t\omega}))$ coincides with $\ker(\bar{\partial} + A_{t\omega})$. ■

The following claim gives a description of the complex structure associated with $\bar{\partial}_{new} = \bar{\partial} + tI \circ \partial$, in terms of the hyperkähler structure.

Claim 11.7: Let M be a hyperkähler manifold and

$$\bar{\partial}_{new} = \bar{\partial} + tJ \circ \partial : C^\infty \longrightarrow \Lambda^1(M)$$

be the operator considered above. Then, there exists a unique induced² complex structure L , and an invertible element $\alpha \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, such that $\bar{\partial}_L = \alpha(\bar{\partial} + tJ \circ \partial)$.

Proof: Let

$$\mathcal{D} = \text{Diff}^1(C^\infty(M), \Lambda^1(M))$$

be the space of all differential operators of first order from $C^\infty(M)$ to $\Lambda^1(M)$. The space \mathcal{D} is equipped with a left action of the quaternions, induced by the action of the quaternions in $\Lambda^1(M)$. Let $D \subset \mathcal{D}$ be the subspace of \mathcal{D} generated by $\bar{\partial}_L$, $\partial_L = \bar{\partial}_{-L}$, for all induced complex structures L . Clearly, D contains the standard de Rham differential $d = \frac{\partial_L + \bar{\partial}_L}{2}$. Let $d_L^c := \frac{\partial_L - \bar{\partial}_L}{2\sqrt{-1}}$ be the de Rham differential twisted by L , which also lies in D . It is easy to check that d, d_I^c, d_J^c, d_K^c constitute a basis in D . We identify D with $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, with d going to $1 \in \mathbb{H}$, and d_α^c going to α , $\alpha = I, J, K \in \mathbb{H}$. Clearly, D is preserved by the left action of the quaternions on \mathcal{D} , and is naturally isomorphic to the space to $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \text{Mat}_{\mathbb{C}}(2)$ as a representation

²As usual, **induced complex structure** means a complex structure L induced by a hyperkähler structure, $L = aI + bJ + cK$, $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$.

of $(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})^* = GL(2, \mathbb{C})$. Let $C_0 \subset Mat_{\mathbb{C}}(2)$ be the set of all $x \neq 0$ with $\det x = 0$. Under the identification $D \xrightarrow{i} Mat_{\mathbb{C}}(2)$, the operators ∂_L go to C_0 , for all induced complex structures L . Also, $i(\bar{\partial} + tJ \circ \partial)$ belongs to C_0 . The quotient of C_0 by the left action of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$ is $\mathbb{C}P^1$, and this $\mathbb{C}P^1$ is in a natural bijective correspondence with the set of induced complex structures on M . Using Claim 11.6, we obtain Claim 11.7. ■

We have reduced Theorem 11.3 to the following statement.

Lemma 11.8: Let M be a hyperkähler manifold, and L be the induced complex structure constructed in Claim 11.7. Then the 2-form $\Omega + 2t\omega + t^2\bar{\Omega}$ is of type $(2,0)$ with respect to L .

Proof: As with Claim 11.7, this statement is proven by an elementary linear-algebraic computation. Let $W \subset \Gamma_M(\Lambda^2(M))$ be the \mathbb{C} -linear space generated by $\omega_I, \omega_J, \omega_K$. Consider the standard action of $SU(2)$ on W . We obtain a representation

$$\rho : SU(2) \longrightarrow End(W).$$

Consider the space D of differential operators generated by $\bar{\partial}_L$ for all induced complex structures (see Claim 11.7, Proof). Then D is canonically isomorphic to $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. The group $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^* = GL(2, \mathbb{C})$ acts on $C_0 \subset D = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ from the left and from the right, with the left action being the same as we considered in Claim 11.7 Proof. We have identified the left quotient $((\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})^*) \backslash C_0$ with the set \mathcal{R} of all induced complex structures. Let $\mathbb{H}^{un} = SU(2)$ be the group of unitary quaternions. Clearly, the right action of $\mathbb{H}^{un} \subset \mathbb{H}^* \subset \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$ on D induces the natural action of $\mathbb{H}^{un} = SU(2)$ on $\mathbb{C}P^1 = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^* \backslash C_0$. In particular, under the right action of $\mathbb{H}^{un} \subset \mathbb{H}^* \subset \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$ on C_0 , we have

$$g(\partial_L) = \partial_{g(L)}, \quad \text{and} \quad g(\bar{\partial}_L) = \bar{\partial}_{g(L)} \quad (11.7)$$

where $L \in \mathcal{R}$ is the induced complex structure, and $g(L)$ is an image of L under the natural action of $\mathbb{H}^{un} = SU(2)$ on $\mathcal{R} = \mathbb{C}P^1$.

Let $P_c : \mathcal{R} \longrightarrow \mathbb{P}(W)$ be the map associating with an induced complex structure L the line of all forms $\lambda \in W$ which are of type $(2,0)$ with respect to L . Clearly, for all $g \in SU(2) \otimes \mathbb{C}$, $L \in \mathcal{R}$, we have

$$\rho(g)(P_c(L)) = P_c(g(L)), \quad (11.8)$$

where $\rho : SU(2) \longrightarrow \text{End}(W)$ is the representation defined above. The space W is equipped with a natural Hermitian metric $(\cdot, \cdot)_{\mathcal{H}}$, such that $\omega_I, \omega_J, \omega_K$ is an orthonormal basis. Whenever $\lambda \in W$ is a $(2, 0)$ -form for some induced complex structure, we have $(\lambda, \lambda)_{\mathcal{H}} = 0$. Let $C \subset \mathbb{P}(W)$ be the conic defined by $(\lambda, \lambda)_{\mathcal{H}} = 0$. This conic is naturally isomorphic to $\mathbb{C}P^1$, so that the map $P_c : \mathcal{R} \longrightarrow C$ is an isomorphism. Consider the Lie algebra $\mathfrak{g}_0(\mathcal{H}) \subset \text{End}(\Lambda^2(M))$ generated by $[L_{\omega_\alpha}, \Lambda_{\omega_\beta}]$, for $\alpha, \beta \in \{I, J, K\}$, $\alpha \neq \beta$. From (8.2) it is clear that $\mathfrak{g}_0(\mathcal{H})$ is naturally isomorphic to $\mathfrak{su}(2)$. Let $G_0(\mathcal{H}) = SU(2)$ be the corresponding Lie group, acting on the cohomology of M . Using the identification $G_0(\mathcal{H}) \otimes \mathbb{C} = SU(2) \otimes \mathbb{C} = GL(2, \mathbb{C})$, we may consider $e^{t[L_{\omega_\alpha}, \Lambda_{\omega_\beta}]}$ as an element of $SU(2) \otimes \mathbb{C} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$. Let $p \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$ be the element which corresponds to $e^{t[L_{\omega}, \Lambda_{\bar{\Omega}}]}$. By definition,

$$\rho(p)(\Omega) = \Omega + 2t\omega + t^2\bar{\Omega}.$$

On the other hand, a calculation shows that, under the natural right action of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$ on $D = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, we have $p(\bar{\partial}) = \bar{\partial} + tJ\partial$. By (11.7), (11.8), the vector

$$\rho(p)(\Omega) = \Omega + 2t\omega + t^2\bar{\Omega}$$

belongs to the line $P_c(L)$. This implies that, for the induced complex structure L of Claim 11.7, the differential operators $p(\bar{\partial})$ and $\bar{\partial}_L$ define the same complex structure, where $p(\bar{\partial})$ means the image of $\bar{\partial} \in D$ under the *right* action of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^*$. Lemma 11.8, and consequently, Theorem 11.3 is proven. \blacksquare

12 Proof of Mirror Conjecture.

In this section, we prove Theorem 5.4. Let \mathcal{B}_1 be an open ball in $H^{n-1,1}(M)$, $\varphi : \mathcal{B}_1 \longrightarrow \text{Comp}$ the Tian–Todorov map, $I = \varphi(0)$. Let \mathcal{B}_2 be an open ball in $H^{1,1}(M)$. We identify \mathcal{B}_1 with its image under φ , considering \mathcal{B}_1 as an open subset in Comp . We denote by \mathcal{A}_1 the Yukawa VFA associated with a trivialization of $K|_{\varphi(\mathcal{B}_1)}$ which we shall specify soon thereafter. Let \mathcal{A}_2 be a Quantum VFA, with base \mathcal{B}_2 . Shrinking $\mathcal{B}_1, \mathcal{B}_2$ when necessary and using the natural isomorphism $H^{1,1}(M) \cong H^{n-1,1}(M)$, we obtain a canonical linear isomorphism

$$t : \mathcal{B}_1 \longrightarrow \mathcal{B}_2. \tag{12.1}$$

To prove Theorem 5.4 we need to produce an isomorphism of VFA $\theta : \mathcal{A}_1 \longrightarrow t^* \mathcal{A}_2$, where $t^* \mathcal{A}_2$ is the pullback of \mathcal{A}_2 under t .

Let C be, as usual, the quadric hypersurface defined by $(x, x)_{\mathcal{H}} = 0$, and $P_c : \text{Comp} \longrightarrow C$ be the period map. Consider the restriction $\mathcal{O}(1)|_C$ of $\mathcal{O}(1)$ to $C \subset \mathbb{P}(H^2(M, \mathbb{C}))$. Let Ω_s be the holomorphic line bundle on Comp , $\Omega_s = P_c^* \left(\mathcal{O}(1)|_C \right)$. Fix a holomorphic symplectic form Ω on the complex manifold (M, I) . We define a trivialization of the 1-dimensional complex space $\Omega_s|_{\varphi(\mathcal{B}_1)}$ as follows. Let $\alpha \in \mathcal{B}_1$. Consider α as a $(1, 1)$ -form. By definition, the line $P_c(\varphi(\alpha))$ contains a vector $e^{[L_\alpha, \Lambda_\Omega]} \Omega \in H^2(M, \mathbb{C})$, where $e^{[L_\alpha, \Lambda_\Omega]} \in \text{End}(H^*(M))$ is the element of $G_0^{I, 2}(M)$ corresponding to $\alpha \in H^{1, 1}(M)$. Therefore, we may consider $e^{[L_\alpha, \Lambda_\Omega]} \Omega$ as a vector of $\Omega_s|_{\varphi(\alpha)}$. This gives a vector in the fiber of Ω_s for every point $\varphi(\alpha) \in \varphi(\mathcal{B}_1)$ and trivializes Ω_s over $\varphi(\mathcal{B}_1) \subset \text{Comp}$. Denote the section $\varphi(\alpha) \longrightarrow e^{[L_\alpha, \Lambda_\Omega]} \Omega \in \Omega_s|_{\varphi(\alpha)}$ by ν . Since $\Omega_s^{n/2} = K$, ν gives a trivialization of $K|_{\varphi(\mathcal{B}_1)}$. By \mathcal{A}_1 we understand Yukawa VFA associated with this trivialization.

For the rest of this section, we denote the endomorphism $e^{[L_\alpha, \Lambda_\Omega]} : H^*(M) \longrightarrow H^*(M)$ by e_α . The fibers of $\mathcal{A}_1, \mathcal{A}_2$ are naturally identified with the total cohomology space $H^*(M)$. Let

$$\theta_0 : \mathcal{A}_1 \longrightarrow \mathcal{A}_1$$

be the endomorphism acting on the fiber $\mathcal{A}_1|_{\varphi(\alpha)}$ by $e_{-\alpha}$. Let

$$\eta_0 : \mathcal{A}_1 \longrightarrow t^* \mathcal{A}_2$$

be the constant map acting on the fibers of $\mathcal{A}_1, \mathcal{A}_2$ by

$$\eta_{I, \Omega} : H^*(M) \longrightarrow H^*(M),$$

where $\eta_{I, \Omega}$ is the Serre duality operator associated with (I, Ω) (Section 9). Let

$$\theta = \theta_0 \circ \eta_0 : \mathcal{A}_1 \longrightarrow t^* \mathcal{A}_2.$$

The following theorem proves the Mirror Conjecture for manifolds of hyperkähler type (Theorem 5.4).

Theorem 12.1: The map θ induces an isomorphism of VFA.

Proof: By Proposition 9.7, we have $\eta_0[L_\alpha, \Lambda_\Omega]\eta_0^{-1} = L_\alpha$. Therefore, $\eta_0 e_\alpha \eta_0^{-1}$ is the map

$$\begin{aligned} \cdot \cup e^\alpha : H^*(M) &\longrightarrow H^*(M), \\ \eta_0 e_\alpha \eta_0^{-1}(x) &= x \cup e^\alpha \\ x \cup e^\alpha &= \sum \frac{x \cup \alpha^n}{n!}. \end{aligned} \tag{12.2}$$

Let ∇_m be the flat connection in \mathcal{A}_2 associated with the VFA structure, and $\gamma : \mathcal{B}_2 \longrightarrow H^*(M)$ be an arbitrary map considered as a section of \mathcal{A}_2 . Then γ is parallel with respect to ∇_m if and only if $\gamma(\alpha) = x \cup e^\alpha$, for some $x \in H^*(M)$, because the operator $x \longrightarrow x \cup e^{-\alpha}$ maps ∇_m to the constant connection ∇ (see Corollary 6.2). The operator $\theta = \eta_0 \circ e_{-\alpha}$ maps the constant sections y of \mathcal{A}_1 to the sections $\gamma(\alpha) = \eta_0(y) \cup e^\alpha$, because $\eta_0 e_\alpha(y) = (\eta_0)^{-1}(y) \cup e^\alpha$ by (12.2). In other words, θ maps constant sections of \mathcal{A}_1 to the constant sections of $t^*\mathcal{A}_2$. We obtain the following statement.

Lemma 12.2: The map $\theta : \mathcal{A}_1 \longrightarrow t^*\mathcal{A}_2$ commutes with the flat connection on the variations of Frobenius algebras $\mathcal{A}_1, t^*\mathcal{A}_2$.

■

To prove that θ is compatible with the Hodge filtration, we make the following observation (Lemma 12.4).

As in (11.1), consider the decomposition

$$\mathfrak{g}_0(M) \otimes \mathbb{C} = \mathfrak{g}_0^{I,-2}(M) \oplus \mathfrak{g}_0^{I,0}(M) \oplus \mathfrak{g}_0^{I,2}(M). \tag{12.3}$$

Let $G_0^{I,-2}(M), G_0^{I,0}(M), G_0^{I,2}(M) \subset \text{End}(H^*(M))$ be the corresponding Lie groups.

Remark 12.3: The multiplication

$$G_0^{I,-2}(M) \times G_0^{I,0}(M) \times G_0^{I,2}(M) \longrightarrow G_0(M)$$

is surjective in a neighbourhood of unit in $G_0(M)$.

Lemma 12.4: For a complex structure $L \in \text{Comp}$, consider the Hodge filtration $F_L^0 \subset F_L^1 \subset \dots$ on $H^p(M)$,

$$F_L^k := \bigoplus_{i-q \leq k} H^{p-i,q}(M). \quad (12.4)$$

associated with L . Then $e_{-\alpha} : H^p(M) \longrightarrow H^p(M)$ maps $F_{\varphi(\alpha)}^i$ to F_I^i .

Proof: Consider the standard action of a group $G_0(M)$ on

$$C \subset \mathbb{P}H^2(M, \mathbb{C}).$$

By Theorem 7.4, for $I_1, I_2 \in \text{Comp}$, $g \in G_0(M)$, such that

$$P_c(I_1) = g(P_c(I_2)),$$

we have

$$H_{I_1}^{p,q}(M) = g(H_{I_2}^{p,q}(M)).$$

Since $G_0(M)$ acts transitively on C , there exists $g \in G_0(M)$ such that

$$g(H_I^{p,q}(M)) = H_{\varphi(\alpha)}^{p,q}(M).$$

Shrinking \mathcal{B}_1 if necessary, we may assume that g admits the decomposition considered in Remark 12.3, $g = g_{-2}g_0g_2$, with $g_i \in G_0^{I,i}(M)$, $i = -2, 0, 2$. Under the natural action of $G_0(M) \otimes \mathbb{C}$ on $C \subset \mathbb{P}(H^2(M, \mathbb{C}))$, we have $g(P_c(I)) = P_c(\varphi(\alpha))$. Since $P_c(\varphi(\alpha))$ is by definition a line which passes through $e_\alpha(\Omega)$, we obtain that $g(\Omega)$ is proportional to $e_\alpha(\Omega)$. The group $G_0^{I,2}(M)$ stabilizes Ω , and $G_0^{I,0}(M)$ stabilizes the line passing through Ω . Therefore, $g_{-2}(\Omega)$ is proportional to $e_\alpha(\Omega)$. The map $G^{I,-2}(M) \longrightarrow C$, $g \longrightarrow g(\Omega)$ is locally an isomorphism (see Section 11). Therefore, $g_{-2} = e_\alpha$. By Claim 10.5, the operator g maps the Hodge grading associated with the complex structure I to the Hodge grading associated with $\varphi(\alpha)$. To prove that the operator e_α is compatible with Hodge filtration, we need to prove that $g_0g_2 = (g_{-2})^{-1}g$ preserves the Hodge filtration associated with I . This is implied by the following general statement.

Sublemma 12.5: Let M be a compact holomorphically symplectic manifold of Kähler type, I its complex structure, $G_0(M) \subset \text{End}(H^*(M))$ be the standard group acting on its cohomology, $G_0^{I,-2}(M)$, $G_0^{I,0}(M)$, $G_0^{I,2}(M)$ be subgroups of $G_0(M) \otimes \mathbb{C}$ associated with I and a decomposition (12.3), and s be any element of $G_0^{I,0}(M) \times G_0^{I,2}(M)$. Then the action of s on $H^*(M)$ preserves the Hodge filtration (12.4) associated with the complex structure I .

Proof: By definition, $G_0^{I,0}(M)$ preserves the Hodge grading $H^*(M) = \oplus H_I^{p,q}(M)$. Therefore, $G_0^{I,0}(M)$ preserves the Hodge filtration. It remains to show that $G_0^{I,2}(M)$ preserves the Hodge filtration. The group $G_0^{I,2}(M)$ is by definition connected, and therefore we have to show that for all $\mu \in \mathfrak{g}_0^{I,2}(M)$, the action of μ on $H^*(M)$ preserves Hodge filtration. By definition, $[\mu, ad I] = -2\sqrt{-1} ad I$. Therefore, $ad I(\mu(\omega)) = (i-2)\sqrt{-1}\mu(\omega)$, for all $\omega \in H^{p+i,p}(M)$. This means that $\mu(\omega) \in H^{p+i-1,p+1}(M)$. Sublemma 12.5 and consequently Lemma 12.4 is proven. ■

Remark 12.6: The group $G_0^{I,2}(M)$ manifestly does not preserve the Hodge grading. Hence (Sublemma 12.5) is false if we substitute “filtration” by “grading”.

Corollary 12.7: The map $\theta : \mathcal{A}_1 \longrightarrow t^*\mathcal{A}_2$ is compatible with Hodge filtration associated with the VFAs $\mathcal{A}_1, \mathcal{A}_2$.

Proof: By definition of Serre duality, the map η_0 maps the Hodge filtration in $\mathcal{A}_1|_{\varphi(0)}$ to the Hodge filtration in $\mathcal{A}_2|_0$. Since the Hodge filtration in \mathcal{A}_2 is constant with respect to the trivial connection ∇_0 , we can use Lemma 12.4 and obtain Corollary 12.7. ■

Let $\mathcal{A}_1^{gr}, \mathcal{A}_2^{gr}$ be the associated graded bundles for the variations of Frobenius algebras $\mathcal{A}_1, \mathcal{A}_2$. The bundles $\mathcal{A}_1^{gr}, \mathcal{A}_2^{gr}$ are naturally equipped with a structure of weak VFA. By Corollary 12.7, the map θ induces a map of associated graded spaces $\theta^{gr} : \mathcal{A}_1^{gr} \longrightarrow t^*\mathcal{A}_2^{gr}$. To finish the proof of Theorem 12.1, we need to show only that θ^{gr} is compatible with the weak VFA structure.

Consider the maps $\kappa_i : \mathcal{A}_i^{gr} \otimes T\mathcal{B}_i \longrightarrow \mathcal{A}_i^{gr}$, $\kappa(a \otimes t) = a \bullet t$, where \bullet is the multiplication in VFA and τ_i is the standard map which comes with the definition of VFA. By definition, κ_i is the Kodaira-Spencer map associated with a weak \mathbb{C} -VHS. Since θ commutes with the weak \mathbb{C} -VHS structure (Corollary 12.7, Lemma 12.2), the following diagram is commutative:

$$\begin{array}{ccc}
 T\mathcal{B}_1 \otimes \mathcal{A}_2^{gr} & \xrightarrow{\kappa_1} & t^*\mathcal{A}_2^{gr} \\
 dt \otimes \theta^{gr} \downarrow & & \downarrow \theta^{gr} \\
 t^*T\mathcal{B}_2 \otimes \mathcal{A}_1^{gr} & \xrightarrow{t^*\kappa_2} & t^*\mathcal{A}_1^{gr}
 \end{array} \tag{12.5}$$

Assuming that θ^{gr} is an algebra homomorphism, we obtain that the following

diagram is also commutative:

$$\begin{array}{ccc}
 T\mathcal{B}_1 & \xrightarrow{\tau_1} & t^*\mathcal{A}_2^{gr} \\
 dt \downarrow & & \downarrow \theta^{gr} \\
 t^*T\mathcal{B}_2 & \xrightarrow{t^*\tau_2} & t^*\mathcal{A}_1^{gr}
 \end{array} \tag{12.6}$$

Therefore, to finish the proof of Theorem 12.1, we need only prove the following proposition.

Proposition 12.8: Let $\theta^{gr} : \mathcal{A}_1^{gr} \longrightarrow t^*\mathcal{A}_2^{gr}$ be the map constructed above. Then θ^{gr} is compatible with the multiplicative structure in \mathcal{A}_1^{gr} , \mathcal{A}_2^{gr} .

Proof: Consider the usual cohomology algebra $(H^*(M), \cup)$ of M . Then \mathcal{A}_2^{gr} is a trivial bundle with a fiber $(H^*(M), \cup)$, and the multiplication in \mathcal{A}_2^{gr} is compatible with this trivialization. By definition of the Serre duality operator, the map $\eta_0 : \mathcal{A}_1|_{\varphi(0)} \longrightarrow (H^*(M), \cup)$ commutes with the algebraic structure. Therefore, to prove Proposition 12.8, we have to show that the map¹ $e_{-\alpha} : \mathcal{A}_1|_{\varphi(\alpha)} \longrightarrow \mathcal{A}_1|_{\varphi(0)}$ induces a homomorphism of algebras

$$e_{-\alpha}^{gr} : \left(\mathcal{A}_1|_{\varphi(\alpha)} \right)^{gr} \longrightarrow \left(\mathcal{A}_1|_{\varphi(0)} \right)^{gr}.$$

Consider the natural action of the group $G_0(M)$ on $H^*(M)$. This action induces an action of $G_0(M)$ on the tensor powers

$$(H^*(M))^{\otimes n} \otimes ((H^*(M))^*)^{\otimes n}.$$

Denote this action by $\lambda \longrightarrow (\lambda)^g$. Let $I = \varphi(0)$, $J = \varphi(\alpha)$, and $\bullet_I, \bullet_J : H^*(M) \times H^*(M) \longrightarrow H^*(M)$ be the Yukawa product maps in $\mathcal{A}_1|_I, \mathcal{A}_1|_J$. Let $\cup : H^*(M) \times H^*(M) \longrightarrow H^*(M)$ be the usual cup-product. Consider $\bullet_I, \bullet_J, \cup$ as tensors over the space $H^*(M)$. By (9.1), we have

$$(\bullet_I) = (\cup)^{\eta_0}, \quad (\bullet_J) = (\cup)^{\eta_\alpha}, \tag{12.7}$$

where

$$\eta_0, \quad \eta_\alpha$$

¹By Lemma 12.4, $e_{-\alpha}$ is compatible with the Hodge filtrations on $\mathcal{A}_1|_{\varphi(\alpha)}, \mathcal{A}_1|_{\varphi(0)}$, associated with VFA.

are the Serre duality operators associated with $(\varphi(0), \Omega)$ and $(\varphi(\alpha), e_\alpha(\Omega))$. By definition, $\eta_\alpha = (\eta_0)^g$, where $g \in G_0(M)$ is the group element which defined in the proof of Lemma 12.4. Therefore, (12.7) implies that

$$(\bullet_J) = (\bullet_I)^g.$$

Consider a decomposition $g = g_2 g_0 g_{-2}$ defined in the proof of Lemma 12.4. As we have seen, $e_{-\alpha} = g^{-1} g_0 g_2$. The map

$$g^{-1} : (H^*(M), \bullet_J) \longrightarrow (H^*(M), \bullet_I)$$

induces an isomorphism of Frobenius algebras and is compatible with the Hodge filtration. Hence, g induces an isomorphism on the associated graded algebras. To prove that $(g_2)^{-1} = e_\alpha : (H^*(M), \bullet_J) \longrightarrow (H^*(M), \bullet_I)$ induces an isomorphism on the associated graded algebras, we need to show that

$$g_0 g_2 : (H^*(M), \bullet_I) \longrightarrow (H^*(M), \bullet_I)$$

induces an isomorphism on associated graded algebras. By Theorem 7.3,

$$g_0 : (H^*(M), \bullet_I) \longrightarrow (H^*(M), \bullet_I)$$

is an algebra isomorphism. Since $G_I^{I,0}(M)$ preserves the Hodge grading, g_0 certainly induces an isomorphism on the associated graded algebras. It remains to show that $g_2 : (H^*(M), \bullet_I) \longrightarrow (H^*(M), \bullet_I)$ induces an isomorphism on the associated graded algebras. This is implied by the following little lemma, which finishes the proof of Theorem 12.1.

Lemma 12.9: Let $g_2 \in G_0^{2,I}(M)$ be a group element and

$$g_2 : H^*(M) \longrightarrow H^*(M)$$

be the corresponding endomorphism of $H^*(M)$. Consider the Hodge filtration on $H^*(M)$, defined from the VFA structure on \mathcal{A}_1 and an isomorphism $H^*(M) \cong \mathcal{A}_1|_I$.² Then g_2 acts as the identity on the associated graded space.

Proof: The group $G_0^{2,I}(M)$ is by definition connected. Thus, to prove that g_2 acts as the identity on the associated graded space, we need to show

²By definition of \mathcal{A}_1 , this filtration coincides with the standard Hodge filtration on the cohomology space $H^*(M)$ associated with the complex structure I .

that $\mathfrak{g}_0^{2,I}(M)$ acts trivially on the associated graded space. In other words, for all $\lambda \in \mathfrak{g}_0^{2,I}(M)$, we need to show that

$$\lambda(F_I^i H^*(M)) \subset F_I^{i-1} H^*(M), \quad (12.8)$$

where F_I^i is the Hodge filtration associated with I . By definition of $\mathfrak{g}_0^{2,I}(M)$, for all $\omega \in H_I^{p,q}(M)$, we have $\lambda(\omega) \in H_I^{p+1,q-1}(M)$. This proves (12.8). Lemma 12.9 is proven. We have finished the proof of Mirror Symmetry for compact holomorphically symplectic manifolds of Kähler type. ■

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References

- [AM] P. S. Aspinwall and D. R. Morrison, String Theory on K3 Surfaces, in Essays in Mirror Symmetry II, ed. B. Greene, International Press, 1995 (to appear).
- [Beau] Beauville, A. Varietes Kähleriennes dont la premiere classe de Chern est nulle. // J. Diff. Geom. 18, p. 755-782 (1983).
- [BCOV] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Comm. Math. Phys. **165** (1994) pp. 311–427
also:
M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Holomorphic anomalies in topological field theories (with an appendix by S. Katz), Nuclear Phys. B **405** (1993), 279–304.
a beautiful exposition to the above results is found in
M. Bershadsky, V. Sadov, Theory of Kähler Gravity. // preprint, 1994.

- [Bes] Besse, A. Einstein Manifolds // Springer-Verlag, New York (1987)
- [B1] F. A. Bogomolov, Hamiltonian Kähler manifolds, Dokl. Akad. Nauk SSSR **243** (1978), no. 5, 1101–1104, English translation: Soviet Math. Dokl. **19** (1978), 1462–1465.
- [B2] Bogomolov, F. A., Kähler manifolds with trivial canonical class, preprint IHES, Dec 1981 pp. 1-32
- [BB] F. A. Bogomolov, P. Braam, Instantons and Mirror K3 surfaces, // Mathematical Physics 143 (1992) pp. 641-646.
- [BLie] Bourbaki, Nicolas. Groupes et algebres de Lie. // Paris; New York : Masson, 1981.
- [GH] Griffiths Ph. and Harris J., Principles of algebraic geometry. // Wiley-Interscience, New York (1978).
- [KS] Kodaira K., Spencer D. C. On deformations of complex structures II // Kodaira K., Collected Works, vol. II, Princeton Univ Press (1975).
- [KM] M. Kontsevich, Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. // hep-th electronic preprint 9402147, 44 p., also in: Comm. Math. Phys. **164** (1994) pp. 525-562
- [Ko] M. Kontsevich, Homological algebra of mirror symmetry. // alg-geom electronic preprint 9411018, 19 pp.
- [Koz] Koszul, J. L., “Crochet de Schouten-Nijenhuis et cohomologie”. Astérisque (hors série) (1985), 257–271.
- [Li] Lieberman, D., Compactness of the Chow scheme. // Springer Lect. Notes in Math., vol. 670, pp 140-186., (1978)
- [LL] Looijenga, E., Lunts, V. // preprint, 1994.
- [Mrr] Essays in Mirror Symmetry II, ed. B. Greene, International Press, 1995 (to appear).
- [Ti] Tian, G., “Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Petersson-Weil Metric,” in *Mathematical Aspects of String Theory*, 629-646, ed. S.-T. Yau, World Scientific, Singapore, 1987.
- [Tod1] Todorov, A., “The Weil-Petersson Geometry of the Moduli Space of $SU(n \geq 3)$ (Calabi-Yau) Manifolds I,” *Commun. Math. Phys* **126**, (1989) 325-346.
- [Tod2] Todorov A., Moduli of Hyper-Kählerian manifolds I,II. // Preprint MPI (1990)
- [Tod3] Todorov A., Mirror Symmetry for K3 surfaces. // Preprint, 1994.
- [V] Verbitsky, M., Cohomology of compact hyperkähler manifolds. // alg-geom electronic preprint 9501001, 89 pages, LaTeX
- [V-a] Verbitsky, M., Cohomology of compact hyperkähler manifolds and its applications. // alg-geom electronic preprint 9511009, 12 pages, LaTeX.

- [V-so] Verbitsky, M. On the action of a Lie algebra $SO(5)$ on the cohomology of a hyperkähler manifold. // Func. Analysis and Appl. 24(2) p. 70-71 (1990).
- [V-Sym] Verbitsky, M., Hyperkähler embeddings and holomorphic symplectic geometry II. // alg-geom electronic preprint 9403006, 14 pages, LaTeX. // also published in: GAFA vol. 5 no. 1 (1995) pp. 92-104
- [VO] A.L. Onishchik, E.B. Vinberg, Lie groups and Lie algebras III: structure of Lie groups and Lie algebras. // Encyclopaedia of mathematical sciences, v. 41, Springer-Verlag, 1994.
- [Y] Yau, S. T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. // Comm. on Pure and Appl. Math. 31, 339-411 (1978).